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Model order reduction with preservation of passivity, non-expansivity and Markov moments

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ABSTRACT

A new model order reduction technique is presented which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of linear matrix inequalities to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the linear matrix inequality can be replaced by equivalently solving an algebraic Riccati equation, which is known to be a more efficient approach. A new algebraic Riccati equation and a frequency inversion technique are also presented to specifically deal with the important singular case. The preservation of Markov moments is also guaranteed by the judicious choice of a projection matrix. Three pertinent examples comparing the present approach with positive-real balanced truncation show the strength and accuracy of the present approach.

1. Introduction

The use of model order reduction (MOR) aiming at obtaining compact descriptions of initially large linear state space models has become a standard component in computer-aided design methodologies for a large number of engineering and physics applications. For a good introductory textbook on MOR the reader is referred to [1]. Three MOR approaches can currently be distinguished [2]. The first approach consists of the singular value decomposition (SVD) based methods, comprising the balanced realization method [3] and Hankel norm approximation [4]. The second approach consists of the projection-based Krylov-subspace methods [5], comprising the Laguerre-SVD approach [6,7]. The third approach consists of iterative methods combining aspects of both the SVD and Krylov methods [8]. In the excellent overview paper [2] both strengths and weaknesses of the three approaches are analyzed; e.g., the first and third approaches generally preserve stability, while the second approach is fast but does not in general guarantee stability (but see also [7]).

Passivity is an important property to satisfy because stable, but non-passive macro-models can produce unstable systems when connected to other stable, even passive, loads. It is well-known that passivity is equivalent with the positive-realness of the system transfer function. The equivalent form of passivity for a scattering matrix representation is non-expansivity or bounded-realness [9,10]. It is well established that model reduction techniques with preservation of passivity mostly belong to the balanced truncation class [11–14] or are spectral interpolation-based methods [15–17]. In the case of projection-based Krylov methods the problem of preservation of passivity has been studied by several researchers; for an overview of existing approaches see [18,19,6,20–22]. The problem with the Krylov-based passivity preserving methods is that they often assume a special descriptor state space setting that may not always be feasible [12]. For Krylov subspace methods such as PRIMA [21] to generate a passive reduced order model, it is well known [12] that the system must be in a special descriptor state space form, induced by the so-called modified nodal analysis representation [21] of passive networks. Otherwise PRIMA will generate a not necessarily passive reduced order model.

In this paper, we present a new passivity-preserving and non-expansivity-preserving MOR technique, which does not require any special internal structure of the state space model. It is a projection-based method which exploits the solution of linear matrix inequalities (LMI's) to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation (ARE), which is known to be a more efficient approach [23,24].

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While the LMI solvers of [23] are significantly faster than classical convex optimization algorithms, the complexity of LMI computations can grow quickly with the number of states n. For example, the number of operations required to solve a Riccati equation is $O(n^3)$, while the cost of solving an equivalent Riccati inequality LMI [24] is $O(n^6)$. Of course, for large-scale problems, the $O(n^3)$ complexity may still be prohibitive, and in that case fast iterative methods such as the ones in [13,25] may alleviate the cost of solving the Riccati equations.

This paper is organized as follows. Section 2 describes the new technique and contains the proof of its passivity-preserving and non-expansivity-preserving properties. Section 3 deals with the important singular case and presents a new ARE and a frequency inversion technique specifically tailored to the singular case. Section 4 presents pertinent choices for the Krylov projection matrices in such a way that the Markov moments of the system are also preserved. The main novelty of our approach, as compared to positive-real balanced truncation (PRBT) [12,13], is that we only need to solve a single Riccati equation, instead of the two dual Riccati equations in PRBT. Also, while PRBT admits theoretically provable error bounds, which is not the case in the present method, our approach preserves Markov moments or Laguerre expansion coefficients. The present technique could be most adequately described as a hybrid guaranteed passive model order reduction method, preserving most of the benefits of both positive-real balanced truncation and projection-based Krylov subspace methods. Finally, in Section 5 we outline the basics of positive-real balanced truncation, reformulate PRBT in an important singular case, and provide three pertinent examples comparing the present approach with positive-real balanced truncation.

2. Main results

Notation: Throughout the paper X^T and X^H respectively denote the transpose and Hermitian transpose of a matrix X, and I_n denotes the identity matrix of dimension n. For two Hermitian matrices X and Y, the matrix inequalities $X > Y$ or $X \geq Y$ mean that $X - Y$ is respectively positive definite or positive semidefinite. Of course, $X < Y$ or $X \leq Y$ means $Y > X$ or $Y \geq X$. The closed right halfplane $\Re e[s] \geq 0$ is denoted $\mathbb{C}_+$.

2.1. Positive-real systems

For the real system with minimal realization

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where $B \neq 0$, $C \neq 0$ are respectively $n \times p$ and $p \times n$ real matrices and $A \neq 0$ is an $n \times n$ real matrix, to be passive, it is required that the $p \times p$ transfer function

$$H(s) = C(sI_n - A)^{-1}B + D$$

is analytic in $\mathbb{C}_+$, such that

$$H(s) + H(s)^H \geq 0 \quad \forall s \in \mathbb{C}_+.$$

It is well-known [9] that the positive-real lemma in linear matrix inequality (LMI) format: $\exists P^T = P > 0$ such that

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} \leq 0 \tag{2}$$

guarantees the passivity of the system (1). With the additional stronger condition $D + D^T > 0$ (strict passivity at $s = \infty$), the LMI (2) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the algebraic Riccati equation (ARE)

$$A^TP + PA + (PB - C^T)W_p(PB - C^T)^T = 0 \tag{3}$$

where

$$W_p = (D + D^T)^{-1}.$$

The ARE (3) is generally solved by constructing the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A - BW_pC & BW_pB^T \\ -C^TW_pC & -A^T + C^TW_pB^T \end{bmatrix}. \tag{4}$$

Then the system (1) is passive, *i.e.*, the LMI (2) is feasible, if and only if $\mathcal{H}$ has no purely imaginary eigenvalues [26].

Before tackling the main results, we need to define what is meant by a descriptor state space system. It is a more general system described by the differential equations

$$E\dot{x} = Ax + Bu \tag{5a}$$

$$y = Cx + Du \tag{5b}$$

where $E \neq 0$ is an $n \times n$ real matrix called the descriptor. In descriptor state space format the transfer function is given by

$$H(s) = C(sE - A)^{-1}B + D.$$

Note that it is usually required that $sE - A$ is a regular matrix pencil, *i.e.*, $\det(sE - A) = 0$ has a finite number of s values as solutions. When E is singular, the conversion of the descriptor system into a standard state space form can be performed by using the SVD coordinates-based approach [27] or computing a Weierstrass-like form of the pencil matrix [28]. However, since these methods are usually difficult to apply, a more practical approach for dealing with the singular descriptor case is by working implicitly in state space [29,30].

In our case we will only need the simple nonsingular descriptor state space format with E nonsingular.

Next suppose $H(s)$ is passive. The following theorem provides a means to obtain a reduced model which preserves passivity.

Theorem 2.1. *Suppose the system (1) is passive and let $P = P^T > 0$ be a solution of the LMI (2). Let U be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function*

$$H_1(s) = CU(sU^TPU - U^TPAU)^{-1}U^TPB + D$$

is passive.

Proof. It is clear that $H_1(s)$ can be written as

$$H_1(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

$$\tilde{A} = (U^TPU)^{-1}U^TPAU$$

$$\tilde{C} = CU \qquad \tilde{B} = (U^TPU)^{-1}U^TPB.$$

Putting $\tilde{P} = U^TPU$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$\mathcal{L}_1 = \begin{bmatrix} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B} - \tilde{C}^T \\ \tilde{B}^T\tilde{P} - \tilde{C} & -D - D^T \end{bmatrix} = \begin{bmatrix} U^T(A^TP + PA)U & U^T(PB - C^T) \\ (B^TP - C)U & -D - D^T \end{bmatrix}.$$

It is easy to show that the matrix $\mathcal{L}_1$ can be written as

$$\mathcal{L}_1 = \mathcal{E}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} \mathcal{E}$$

where

$$\mathcal{E} = \begin{bmatrix} U & 0_{n\times p} \\ 0_{p\times r} & I_p \end{bmatrix}. \tag{6}$$

By virtue of the LMI (2) we conclude that $\mathcal{L}_1 \leq 0$. □

2.2. Bounded-real systems

For the real system with minimal realization (1) to be non-expansive, it is required that the transfer function $H(s)$ is analytic in $\mathbb{C}_+$ such that

$$H(s)^H H(s) \leq I_p \quad \forall s \in \mathbb{C}_+.$$

In this case (see [9]), it is well-known that the bounded-real lemma in LMI format: $\exists P^T = P > 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I_p \end{bmatrix} \leq 0 \tag{7}$$

guarantees the non-expansivity of the system (1). With the additional stronger product condition $D^T D < I_p$ (strict non-expansivity at $s = \infty$), the LMI (7) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the ARE

$$A^T P + PA + C^T C + (PB + C^T D) W_s (PB + C^T D)^T = 0 \tag{8}$$

where

$$W_s = (I_p - D^T D)^{-1}.$$

The ARE (8) is solved by constructing the associated Hamiltonian matrix

$$\tilde{\mathcal{H}} = \begin{bmatrix} A + BW_s D^T C & BW_s B^T \\ -C^T \tilde{W}_s C & -A^T - C^T D W_s B^T \end{bmatrix} \tag{9}$$

where

$$\tilde{W}_s = (I_p - DD^T)^{-1}.$$

Then the system (1) is non-expansive, *i.e.*, the LMI (7) is feasible, if and only if $\tilde{\mathcal{H}}$ has no purely imaginary eigenvalues [26].

Suppose $H(s)$ is non-expansive. The following theorem provides a means to obtain a reduced model which preserves non-expansivity.

Theorem 2.2. *Suppose the system (1) is non-expansive and let $P = P^T > 0$ be a solution of the LMI (7). Let U be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function*

$$H_2(s) = CU(sU^T PU - U^T PAU)^{-1} U^T PB + D$$

is non-expansive.

Proof. Similar to Theorem 2.1. It is clear that $H_2(s)$ can be written as

$$H_2(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

$$\tilde{A} = (U^T PU)^{-1} U^T PAU$$

$$\tilde{C} = CU \qquad \tilde{B} = (U^T PU)^{-1} U^T PB.$$

Putting $\tilde{P} = U^T PU$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$\mathcal{L}_2 = \begin{bmatrix} \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{C}^T \tilde{C} & \tilde{P}\tilde{B} + \tilde{C}^T D \\ \tilde{B}^T \tilde{P} + D^T \tilde{C} & D^T D - I_p \end{bmatrix}$$
$$= \begin{bmatrix} U^T (A^T P + PA + C^T C) U & U^T (PB + C^T D) \\ (B^T P + D^T C) U & D^T D - I_p \end{bmatrix}.$$

It is easy to show that the matrix $\mathcal{L}_2$ can be written as

$$\mathcal{L}_2 = \mathcal{E}^T \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I_p \end{bmatrix} \mathcal{E}$$

with $\mathcal{E}$ as in (6). By virtue of the LMI (7) we conclude that $\mathcal{L}_2 \leq 0$. □

Remark 2.1. In Section 4 we will show how the projection matrix U can be chosen in order to preserve a selection of the so-called Markov moments of the system.

3. The singular case

In the positive-real case the LMI (2) and the ARE (3) are equivalent only in the case $W_p > 0$ or $D + D^T > 0$. Similarly, in the bounded-real case the LMI (7) and the ARE (8) are equivalent only in the case $W_s > 0$ or $D^T D < I_p$. It is seen that the singular cases $D + D^T$ singular or $I_p - D^T D$ singular cannot easily be solved by means of ARE's (but see also [31,32] for that matter), since the pertinent Hamiltonian matrices are then undefined. On the other hand, LMI's are convex formulations and can always be solved by convex optimization [33], without needing ARE solvers and/or Hamiltonian matrices. However, we will show we can say more under sufficiently general conditions and still use the ARE formalism. Our approach differs considerably from the approaches in [31,32] in that in our method no state space transformations are needed to obtain the ARE's for the singular case. In order to concentrate solely on the positive-real case the interested reader may find two equivalence lemmas relating bounded-real and positive-real cases in the Appendix. Before proceeding to the singular positive-real case, we need two lemmas:

Lemma 3.1. *If $D + D^T \geq 0$ and $\operatorname{rank}(D + D^T) = r < p$ there exists a $p \times p$ orthogonal transformation matrix Γ such that*

$$\Gamma^T (D + D^T)\Gamma = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

where the $r \times r$ matrix R_r is symmetric positive definite. The positive-realness of $\tilde{H}(s) = \Gamma^T H(s)\Gamma$ is not affected by this transformation.

Proof. See [31]. Note that $r = 0$ corresponds to the totally singular case $D + D^T = 0$. □

Lemma 3.2. *Suppose B and C are full rank. Then there exists a matrix $P = P^T > 0$ that satisfies $PB = C^T$ if and only if $CB = B^T C^T > 0$. Furthermore, in that case, all positive definite solutions of $PB = C^T$ are given by*

$$P = C^T (CB)^{-1} C + B_\perp X B_\perp^T$$

where X is an arbitrary $(n - p) \times (n - p)$ positive definite matrix and $B_\perp$ is the orthonormal null space of B.

Proof. See [34]. Note that if $\ker(B) = \{0\}$, which can happen when $p \geq n$, the only solution is $P = C^T (CB)^{-1} C$. □

The next theorem provides an ARE approach for the singular positive-real case.

Theorem 3.1. *Suppose the positive-real singular system is as in Lemma 3.1, i.e.,*

$$D + D^T = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

with R_r positive definite. Then provided the matrix

$$\mathcal{R} = -(C_s AB_s)^T - C_s AB_s - (C_s B_r - B_s^T C_r^T) R_r^{-1} (B_r^T C_s^T - C_r B_s)$$

(constructively defined in the proof below) is symmetric positive definite, there is a positive definite solution P of the composite algebraic Riccati equation (constructively defined in the proof below):

$$A^T P + PA + (PB_r - C_r^T) R_r^{-1} (PB_r - C_r^T)^T$$
$$+ (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0 \tag{10}$$

which also solves the positive-real LMI (2).

Proof. We start with the LMI formulation by means of the Lur'e equations [35]:

$$A^T P + PA = -Q^T Q$$
$$PB - C^T = -Q^T W$$
$$D + D^T = W^T W.$$

Partitioning the matrices B, C, Q and W as

$$B = [B_r, B_s] \qquad C = \begin{bmatrix} C_r \\ C_s \end{bmatrix} \qquad Q = \begin{bmatrix} Q_r \\ Q_s \end{bmatrix}$$
$$W = \begin{bmatrix} W_r & 0 \\ 0 & 0 \end{bmatrix}$$

we can reformulate the Lur'e equations as:

$$A^T P + PA = -Q_r^T Q_r - Q_s^T Q_s \tag{11a}$$
$$PB_r - C_r^T = -Q_r^T W_r \tag{11b}$$
$$PB_s - C_s^T = 0 \tag{11c}$$
$$R_r = W_r^T W_r. \tag{11d}$$

Eliminating Eqs. (11d) and (11b) we obtain

$$A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T = -Q_s^T Q_s \tag{12a}$$
$$PB_s - C_s^T = 0. \tag{12b}$$

If the aim were solely to solve Eq. (12b), we could utilize Lemma 3.2, but in general this will not be sufficient (except when $\ker(B_s) = \{0\}$), since we also need to satisfy Eq. (12a). Also, the matrices B_s and C_s are not necessarily full rank, which limits the application of Lemma 3.2 in this case.

Nevertheless, we proceed by right-multiplying Eq. (12a) with B_s, yielding

$$A^T C_s^T + PAB_s + (PB_r - C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s) = -Q_s^T Q_s B_s. \tag{13}$$

Defining $W_s = Q_s B_s$ and left-multiplying Eq. (13) with B_s^T, we obtain

$$(C_s AB_s)^T + C_s AB_s + (C_s B_r - B_s^T C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s)$$
$$= -W_s^T W_s. \tag{14}$$

Defining

$$\mathcal{V} = B_r^T C_s^T - C_r B_s$$
$$\mathcal{B} = AB_s + B_r R_r^{-1}\mathcal{V}$$
$$\mathcal{C} = -C_s A + \mathcal{V}^T R_r^{-1} C_r$$
$$\mathcal{R} = -(C_s AB_s)^T - C_s AB_s - \mathcal{V}^T R_r^{-1}\mathcal{V}$$

we can rewrite Eqs. (13) and (14) as

$$P\mathcal{B} - \mathcal{C}^T = -Q_s^T W_s$$
$$\mathcal{R} = W_s^T W_s.$$

Assuming $\mathcal{R}$ positive definite, we can write

$$Q_s^T Q_s = (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T$$

yielding the following composite algebraic Riccati equation for P:

$$A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T$$
$$+ (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0. \quad \square$$

Remark 3.1. In the totally singular case $D + D^T = 0$ the Riccati equation becomes

$$A^T P + PA + (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0$$

with

$$\mathcal{B} = AB$$
$$\mathcal{C} = -CA$$
$$\mathcal{R} = -(CAB)^T - CAB.$$

As a last result, which can also help to find the LMI matrix P in the singular case, we have the following:

Theorem 3.2. *Frequency inversion theorem: Let $H(s) = C(sI_n - A)^{-1}B + D$ be minimal and positive-real with A Hurwitz. Then $G(s) = \tilde{C}(sI_n - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ with*

$$\tilde{A} = A^{-1} \qquad \tilde{B} = A^{-1}B \qquad \tilde{C} = -CA^{-1} \qquad \tilde{D} = D - CA^{-1}B$$

is also positive real and admits the same P matrix as $H(s)$.

Proof. It is straightforward to see that when A is Hurwitz, then A^{-1} is also Hurwitz and vice versa. Also, it is simple to see by substitution (see also [36]) that $G(s) = H(1/s)$. By positive-realness, $H(s)$ admits a factorization [35]:

$$H(s) + H(-s)^T = M(-s)^T M(s) \quad \forall s \in \mathbb{C}_+.$$

Since the mapping $s \mapsto 1/s$ is one-to-one in (extended) $\mathbb{C}_+$, it follows that

$$G(s) + G(-s)^T = H(1/s) + H(-1/s)^T$$
$$= M(-1/s)^T M(1/s) \quad \forall s \in \mathbb{C}_+.$$

In other words $G(s)$ is positive-real. To prove it admits the same P as $H(s)$ we write the Lur'e equations

$$A^T P + PA = -Q^T Q$$
$$PB - C^T = -Q^T W$$
$$D + D^T = W^T W.$$

Define $\mathcal{Q} = -QA^{-1}$ and $\mathcal{W} = W - QA^{-1}B$. It is easy to see that

$$\tilde{A}^T P + P\tilde{A} = -\mathcal{Q}^T\mathcal{Q}.$$

Also

$$-\mathcal{Q}^T\mathcal{W} = A^{-T}\left[Q^T W - Q^T QA^{-1}B\right] = P\tilde{B} - \tilde{C}^T$$

and finally

$$\tilde{D} + \tilde{D}^T = \mathcal{W}^T\mathcal{W}. \quad \square$$

Note that $\tilde{D} = H(0)$ and hence Theorem 3.2 maps the positive-realness problem from $s = \infty$ to $s = 0$. Of course it could be that both $H(\infty) + H(\infty)^T$ and $H(0) + H(0)^T$ are singular, in which case Theorem 3.1 or the approaches in [31,32] will provide solutions.

4. Markov moment preservation

In the Section 2 we showed that passivity and non-expansivity can be preserved by introducing a full rank matrix U. In this section we will show how pertinent column-orthogonal projection matrices U can be constructed which also preserve the so-called Markov moments of the system. To see this, we first write the Laurent expansion of

$$H(s) = C(sI_n - A)^{-1}B + D = C(sP + G)^{-1}R + D$$

with $G = -PA$, $R = PB$, in the vicinity of $s = \infty$.

We have

$$H(s) = D + \sum_{k=0}^{\infty}(-1)^k s^{-k-1} C\Omega^k B$$

where $\Omega = -A$. This can be written as

$$H(s) = \sum_{k=-1}^{\infty}(-1)^k s^{-k-1}\mathcal{M}_k.$$

The coefficients $\mathcal{M}_k = C\Omega^k B,\ k \geq 0$ and $\mathcal{M}_{-1} = -D$ are known (up to a sign) as the Markov moments of $H(s)$ at $s = \infty$. Next consider the $n \times r$ Krylov matrix ($r = pq \leq n$)

$$\mathcal{K} = \left[B, \Omega B, \Omega^2 B, \ldots, \Omega^{q-1} B\right]$$

and consider choosing an orthonormal basis for the columns of $\mathcal{K}$, which can be implemented by performing the 'thin' SVD of the Krylov matrix as $\mathcal{K} = U\Sigma V^T$, and where the $n \times r$ matrix U is column-orthogonal. Putting

$$\tilde{P} = U^T P U \qquad \tilde{G} = U^T G U \qquad \tilde{R} = U^T R$$
$$\tilde{C} = CU \qquad \tilde{\Omega} = \tilde{P}^{-1}\tilde{G} \qquad \tilde{B} = \tilde{P}^{-1}\tilde{R}$$

the new Markov moments are given by

$$\tilde{\mathcal{M}}_{-1} = \mathcal{M}_{-1} = -D \qquad \tilde{\mathcal{M}}_k = \tilde{C}\tilde{\Omega}^k\tilde{B} \quad k = 0, 1, \ldots.$$

We are now in a position to prove (see also [37]):

Theorem 4.1. *With the choice of U as above, the Markov moments are equal up to order $q-1$, i.e., $\tilde{\mathcal{M}}_k = \mathcal{M}_k$ for $k = 0, 1, \ldots, q-1$.*

Proof. Since we have constructed an orthonormal basis for the columns of $\mathcal{K}$, we can write $\Omega^k B = UW_k,\ k = 0, \ldots, q-1$, where W_k is an $r \times p$ matrix. Note that we have $R = PB = PUW_0$ and $\tilde{R} = U^T R = U^T PUW_0 = \tilde{P}W_0$ and hence $\tilde{B} = \tilde{P}^{-1}\tilde{R} = W_0$. Next consider the $n \times n$ matrix

$$Z = U\tilde{P}^{-1}U^T G.$$

By induction, it is easy to prove that $Z^k U = U\tilde{\Omega}^k$ for $k = 0, \ldots, q-1$ and hence

$$\tilde{\mathcal{M}}_k = \tilde{C}\tilde{\Omega}^k\tilde{B} = CZ^k UW_0 = CZ^k B \quad k = 0, \ldots, q-1.$$

There remains to prove that $Z^k B = \Omega^k B$ for $k = 0, \ldots, q-1$. This is clearly the case for $k = 0$. Next suppose that $Z^k B = \Omega^k B$ for some k. Then

$$P^{-1} G Z^k B = \Omega^{k+1} B = UW_{k+1}.$$

Pre-multiplying by $U^T P$ yields

$$U^T G Z^k B = U^T PUW_{k+1} = \tilde{P}W_{k+1}$$

or

$$W_{k+1} = \tilde{P}^{-1}U^T G Z^k B$$

and hence

$$Z^{k+1}B = U\tilde{P}^{-1}U^T G Z^k B = UW_{k+1} = \Omega^{k+1}B. \quad \square$$

Recall that by Theorems 2.1 and 2.2, the reduced order model is passive resp. non-expansive, when the original transfer function $H(s)$ is passive resp. non-expansive. Also, one often wishes to have equal Markov moments calculated about another point than infinity, or else to have Markov moments which are coefficients of a Laguerre expansion [6,7]. All these possibilities can be dealt with by transforming the Laplace variable s by means of a real Möbius transformation

$$s = \frac{\alpha u + \beta}{\gamma u + \delta} \qquad \alpha\delta - \beta\gamma \neq 0. \tag{15}$$

The resulting transfer function in the u-domain is

$$(\gamma u + \delta)C\left[u(\alpha P + \gamma G) + (\beta P + \delta G)\right]^{-1} R + D.$$

Now assuming that $\alpha P + \gamma G$ is nonsingular, we can define the matrices

$$\hat{B} = (\alpha P + \gamma G)^{-1} R \qquad \hat{\Omega} = (\alpha P + \gamma G)^{-1}(\beta P + \delta G).$$

After construction of a base $\hat{U}$ of the Krylov matrix

$$\hat{\mathcal{K}} = \left[\hat{B}, \hat{\Omega}\hat{B}, \hat{\Omega}^2\hat{B}, \ldots, \hat{\Omega}^{q-1}\hat{B}\right] = \hat{U}\hat{\Sigma}\hat{V}^T$$

the reduced matrices are now

$$\tilde{P} = \hat{U}^T P\hat{U} \qquad \tilde{G} = \hat{U}^T G\hat{U} \qquad \tilde{R} = \hat{U}^T R \quad \tilde{C} = C\hat{U}.$$

For example, inserting $\alpha = s_0, \beta = \gamma = 1, \delta = 0$ in (15), we in fact perform a Taylor expansion about s_0, as in [38], and inserting $\beta = \alpha, \gamma = -1, \delta = 1$ in (15), boils down to a scaled Laguerre expansion with scaling factor $\alpha > 0$, as in [6,7]. Of course, by Theorems 2.1 and 2.2, passivity and non-expansivity are always maintained.

5. Numerical simulations and comparisons

In this section we present numerical simulations and comparisons with the popular guaranteed passive positive-real balanced truncation algorithm [12,13]. We first give an overview of classic PRBT and its extension to an important singular case.

5.1. Positive-real balanced truncation

In PRBT, we first have to find the unique stabilizing solutions P, Q of the two dual ARE's:

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T = 0 \tag{16a}$$

$$AQ + QA^T + (QC^T - B)(D + D^T)^{-1}(QC^T - B)^T = 0. \tag{16b}$$

Next, we need to find a transformation matrix S such that

$$S^T PS = \Sigma, \qquad S^{-1}QS^{-T} = \Sigma$$

where Σ is a positive definite diagonal matrix. In order to find S, we factorize P, Q as $P = L_o L_o^T$ and $Q = L_c L_c^T$,[1] and the transformation matrix S is then found as $S = L_c V\Sigma^{-1/2}$, where $U\Sigma V^T = L_o^T L_c$ is the singular value decomposition of the product $L_o^T L_c$. The positive real balanced realization is

$$\hat{A} = S^{-1}AS, \qquad \hat{B} = S^{-1}B, \qquad \hat{C} = CS, \qquad \hat{D} = D.$$

Given the order r of the reduced model, the reduced PRBT realization is then

$$A_{\mathrm{red}} = \hat{A}_{11}, \qquad B_{\mathrm{red}} = \hat{B}_1, \qquad C_{\mathrm{red}} = \hat{C}_1, \qquad D_{\mathrm{red}} = \hat{D}$$

obtained by conformal partitioning [12]:

$$\hat{A} = \begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}, \qquad \hat{B} = \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}, \qquad \hat{C} = \begin{bmatrix}\hat{C}_1 & \hat{C}_2\end{bmatrix}.$$

Further details and error bounds regarding positive-real balanced truncation can be found in [12].

The trouble with PRBT is the supposition $D + D^T > 0$, which is never true in modified nodal analysis [13] where $D = 0$. However, in the total singular case $D + D^T = 0$, the Riccati equation solution defined in Theorem 3.1, Remark 3.1 comes to our rescue. Instead of solving the dual Riccati equation (16), we now need to find the unique stabilizing solutions P, Q of the two modified dual ARE's:

$$A^T P + PA + (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0 \tag{17a}$$

$$AQ + QA^T + (Q\mathcal{C}^T - \mathcal{B})\mathcal{R}^{-1}(Q\mathcal{C}^T - \mathcal{B})^T = 0 \tag{17b}$$

where

$$\mathcal{B} = AB, \qquad \mathcal{C} = -CA, \qquad \mathcal{R} = -(CAB)^T - CAB.$$

Finding the positive-real balancing transformation S then proceeds normally as in the nonsingular case. This of course requires $\mathcal{R} = -(CAB)^T - CAB > 0$, but this is always satisfied in the singular simulation examples in Section 5.2.

[1] Note [13] that this may be any decomposition, such as Cholesky or else the square root eigendecomposition, *i.e.*, from the eigendecomposition $P = \Omega\Lambda\Omega^T$, we may take $L_o = \Omega\Lambda^{1/2}$. Since the Cholesky decomposition of nearly singular positive definite matrices is sometimes awkward to obtain, in this paper we opt for the square root eigendecomposition.

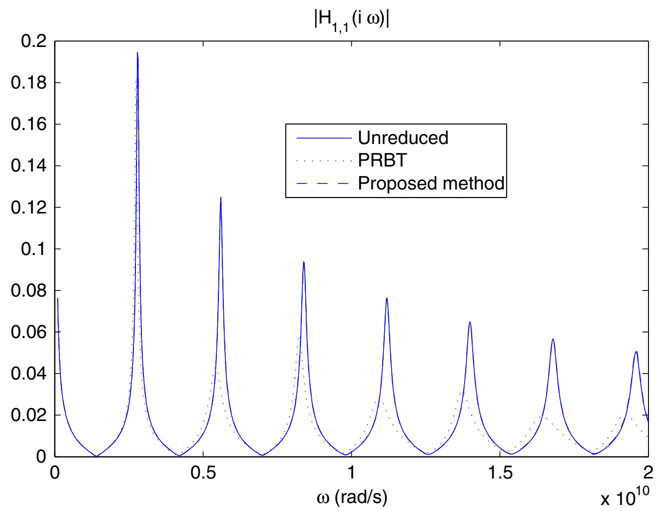

Fig. 1. Bode magnitude plot for the transmission line.

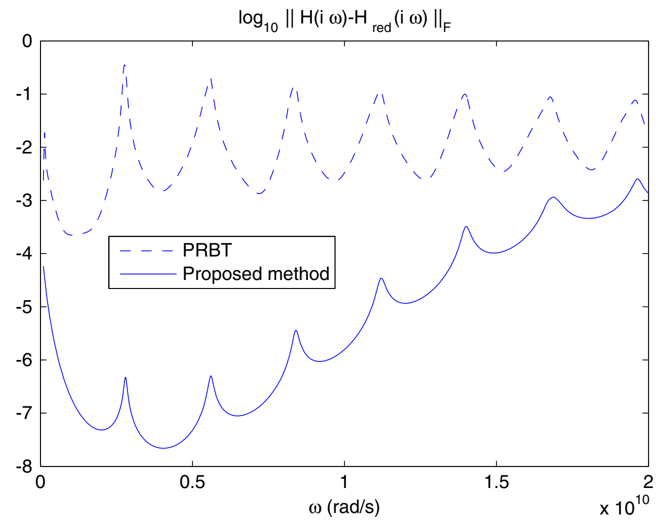

Fig. 2. Frobenius error norm for the transmission line.

5.2. Numerical simulations

In what follows we always compare the method of Theorem 2.1, together with a projection matrix U as in Section 4 — obtained via the scaled Laguerre expansion [6] with scaling factor α — and the PRBT algorithm, potentially modified to treat the singular case as in Section 5.1. As a comparison measure we take the Frobenius norm $\|H(\mathrm{i}\omega) - H_{\mathrm{red}}(\mathrm{i}\omega)\|_F$, calculated over the frequency range of interest.

5.2.1. Transmission line

We consider a passive symmetric two-port transmission line structure with 200 states [39]. The D-matrix is

$$D = \begin{pmatrix} 0.0062 & -0.0001 \\ -0.0001 & 0.0062 \end{pmatrix}$$

and it is easily seen that $D = D^T > 0$. The maximum angular frequency under consideration is $\omega_{\max} = 2 * 10^{10}$ rad/s and the Laguerre scaling factor is $\alpha = \omega_{\max}/2$. The reduced model order is 60. In Fig. 1 we plot the Bode magnitude $|H_{11}(\mathrm{i}\omega)|$ for the original system, the PRBT reduced system and the reduced system obtained by the present method. In Fig. 2 we compare the logarithmic Frobenius error norms $\log_{10} \|H(\mathrm{i}\omega) - H_{\mathrm{red}}(\mathrm{i}\omega)\|_F$ of the proposed method and PRBT. It is seen that the present method is about two orders of magnitude better than PRBT. The processor used was an Intel Core 2Duo E6750/2.66 GHz with 2 GB RAM. The CPU timing was 0.42 s for the Riccati equation (P), 0.40 s for the dual Riccati equation (Q), 0.1 s for the construction of the balancing transformation matrix (S) and 0.05 s for the Laguerre-SVD MOR step. Hence PRBT requires 0.92 s, while the proposed method only requires 0.47 s.

5.2.2. Coupled microstrip lines

Three coplanar microstrips (length $\ell = 20$ cm) over a ground plane with frequency-dependent per-unit-length parameters have been modelled. The cross section is shown in Fig. 3. The conductors have width $w = 100$ μm and thickness $t = 50$ μm. The spacing S between the microstrips is equal to $S = 200$ μm. The dielectric is 300 μm thick and characterized by a dispersive and lossy permittivity which has been modelled by a wideband Debye model [40]. This results in a singular six-port system with 1104 states and $D = 0$. The maximum angular frequency under consideration is $\omega_{\max} = 15 * 10^9$ rad/s and the Laguerre scaling factor is $\alpha = \omega_{\max}/2$. The reduced model order is 500. In Fig. 4 we plot the Bode magnitude $|H_{11}(\mathrm{i}\omega)|$ for the original system, the PRBT reduced system and the reduced system obtained by the present method. In Fig. 5 we compare the logarithmic Frobenius error norms $\log_{10} \|H(\mathrm{i}\omega) - H_{\mathrm{red}}(\mathrm{i}\omega)\|_F$ of the proposed method and singular PRBT. Again it is seen that the proposed method behaves much better than PRBT, except close to the boundaries of the frequency range of interest. The processor used was an Intel Core 2Duo E6750/2.66 GHz with 2 GB RAM. The CPU timing was 9 min 22 s for the Riccati equation, 10 min 1 s for the dual Riccati equation, 10 min 22 s for the construction of the balancing transformation matrix and 7 s for the Laguerre-SVD MOR step. Hence PRBT requires 29 min 45 s, while the proposed method only requires 9 min 29 s.

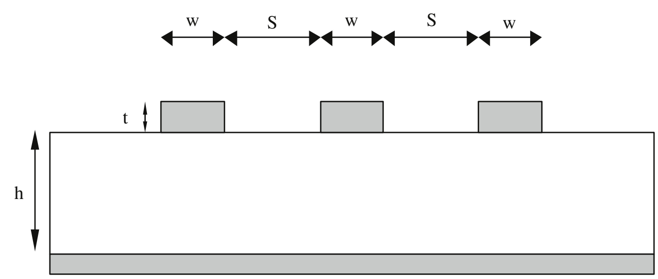

Fig. 3. Cross section of the three coupled microstrips.

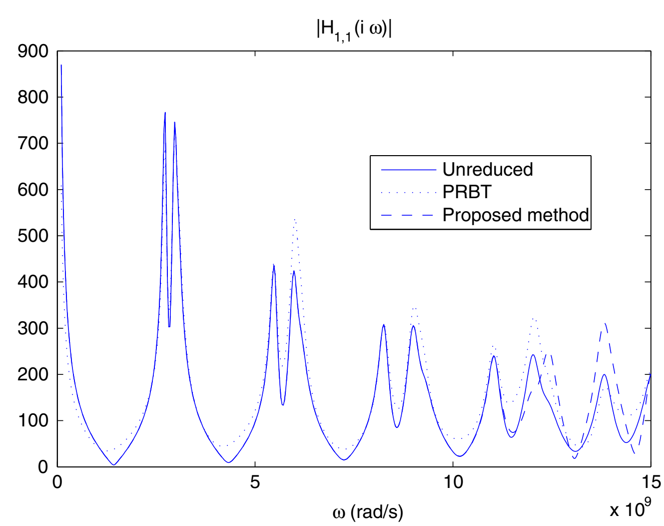

Fig. 4. Bode magnitude plot for the coupled microstrip lines.

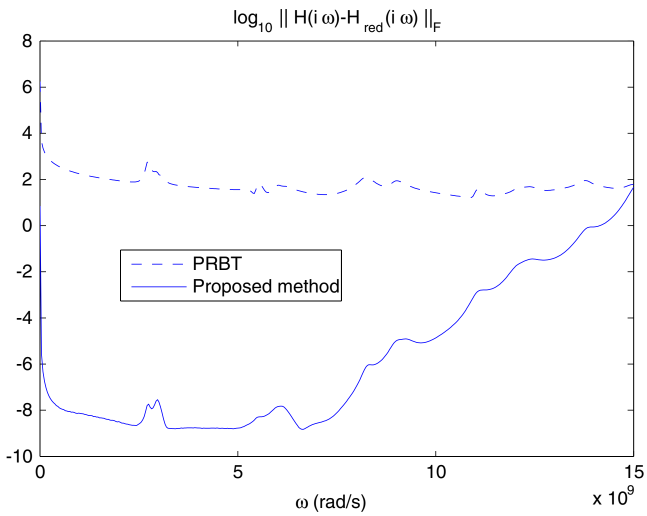

Fig. 5. Frobenius error norm for the coupled microstrip lines.

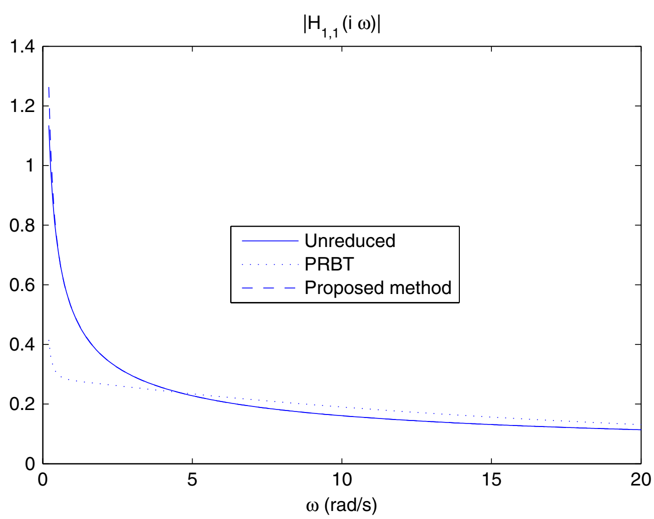

Fig. 6. Bode magnitude plot for the random singular example.

5.2.3. Random singular example

We consider a singular random three-port passive example with 10 000 states and $D = 0$. Note that such a system is easily generated by making use of Theorem 13 in [38], which states that $H(s) = B^T(sE + G)^{-1}B$ is positive real whenever $E = E^T \geq 0$, $G + G^T \geq 0$ and $sE + G$ is a regular matrix pencil. A similar but strictly passive system generator with $D \neq 0$ can be found in [37]. We generated the passive system with the Matlab® code:

```
n=10000; % system order
m=3; % number of ports
epsilon=0.00001; E=randn(n);E=E*E'+epsilon*eye(n); B=randn(n,m);
G=randn(n);G=G*G'; G0=randn(n); G=(G+G')+(G0-G0');
sys=dss(-G,B,B',[],E); [a,b,c,d]=ssdata(sys);
```

The maximum angular frequency under consideration is $\omega_{\max} = 20$ rad/s and the Laguerre scaling factor is $\alpha = \omega_{\max}/2$. The reduced model order is 20. In Fig. 6 we plot the Bode magnitude $|H_{11}(i\omega)|$ for the original system, the PRBT reduced system and the reduced system obtained by the present method. In Fig. 7 we compare the logarithmic Frobenius error norms $\log_{10} \|H(i\omega) - H_{\text{red}}(i\omega)\|_F$ of the proposed method and singular PRBT. It is seen that the proposed method is about two orders of magnitude better than PRBT, except in the low-frequency region. The processor used was a Quad-Core AMD Opteron 2350/8Cores/2.01 GHz with 32 GB

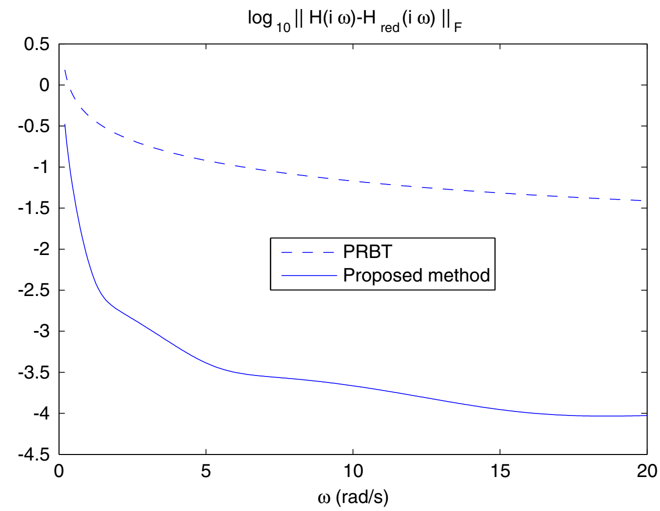

Fig. 7. Frobenius error norm for the random singular example.

RAM. The CPU timing was 12 h 16 min for the Riccati equation, 12 h 49 min for the dual Riccati equation, 8 h 23 min for the construction of the balancing transformation matrix and 3 min 51 s for the Laguerre-SVD MOR step. Hence PRBT requires 33 h 18 min, while the proposed method only requires 12 h 20 min. Note that, once the matrix Riccati equation for P is solved, the remaining MOR step is a mere fraction of the total computational time.

Remark 5.1. Note that, since

$$H(s) = B^T(sE + G)^{-1}B = c(sI - a)^{-1}b + d$$

we have $a = -E^{-1}G$, $b = E^{-1}B$, $c = B^T$, $d = 0$. Hence, most important, it is clear that, given the system matrices a, b, c, there is NO obvious way of recovering the original descriptor state space matrices E, G, B. This implies that, although one can apply Krylov subspace MOR techniques to the descriptor state space format E, G, B, with preservation of passivity—see Corollary 14 in [38]—, this is NOT in general possible with the non-descriptor state space format a, b, c. This of course implies in general that the reduced system will also be in descriptor format, but this need not be a drawback, since what really counts in our approach is the preservation of passivity.

Remark 5.2. From the CPU timings of the random example, it is seen that we are reaching the limits of solving Riccati equations by classical methods. If n significantly exceeds 10^4, then traditional approaches for solving algebraic Riccati equations, such as the Matlab® routine `aresolve`, will fail on a serial computer due to an excessive demand for memory and computing time. There are currently two approaches to remedy this situation: (1) iterative algorithms that exploit sparsity and/or low-rank structure [13,25], and (2) parallel variants of existing serial algorithms [41]. Although the study of the sparse methods (1) is outside the scope of this paper, the fact that the Riccati solution P is typically dense in our case is a major drawback for large-scale application of the proposed method, unless one heavily parallelizes the classical Riccati solvers.

6. Conclusion

We have presented a new model order reduction technique which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of linear matrix inequalities to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the linear matrix inequality can be replaced by equivalently solving an algebraic Riccati equation, which is known to be a faster approach. A new algebraic Riccati equation and a frequency inversion technique are presented to specifically deal with the difficult singular case. We also showed how the pertinent column-orthogonal projection matrix can be constructed such that the Markov moments of the system are also preserved. Finally, three pertinent examples comparing the present approach with positive-real balanced truncation indicate the strength and accuracy of the present approach.

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Appendix

Lemma A.1. *$H(s) = C(sI_n - A)^{-1}B + D$ minimal and bounded-real with A Hurwitz is equivalent with $G(s) = C_1(sI_n - A)^{-1}B + D_1$ positive-real for C_1, D_1 as constructed below.*

Proof. We have

$$I_p - H^T(-s)H(s) \geq 0 \quad \forall s \in \mathbb{C}_+.$$

Now

$$I_p - H^T(-s)H(s) = I - D^TD - (D^TC + B^TW_o)(sI_n - A)^{-1}B - B^T(-sI_n - A^T)^{-1}(C^TD + W_oB)$$

where $W_o > 0$ is the observability Grammian.

Hence taking $D_1 = (I - D^TD)/2$ and $C_1 = -D^TC - B^TW_o$ we see that

$$G^T(-s) + G(s) \geq 0 \quad \forall s \in \mathbb{C}_+. \quad \square$$

Lemma A.2. *If $H(s) = C(sI_n - A)^{-1}B + D$ is minimal and bounded-real such that $\det[I_p - H(s)] \neq 0$ for $\Re e[s] > 0$ then $G(s) = [I_p - H(s)]^{-1}[I_p + H(s)] = \check{C}(sI_n - \check{A})^{-1}\check{B} + \check{D}$ is minimal and positive-real with*

$$\check{A} = A + B(I_p - D)^{-1}C, \qquad \check{B} = \sqrt{2}B(I_p - D)^{-1}$$
$$\check{C} = \sqrt{2}(I_p - D)^{-1}C, \qquad \check{D} = (I_p - D)^{-1}(I_p + D).$$

Conversely, if $G(s) = C(sI_n - A)^{-1}B + D$ is minimal and positive-real then $H(s) = [G(s) - I_p][G(s) + I_p]^{-1} = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ is minimal and bounded-real with

$$\hat{A} = A - B(I_p + D)^{-1}C, \qquad \hat{B} = \sqrt{2}B(I_p + D)^{-1}$$
$$\hat{C} = \sqrt{2}(I_p + D)^{-1}C, \qquad \hat{D} = (D - I_p)(D + I_p)^{-1}.$$

Proof. See [42]. □

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