

Magnitude Vector Fitting to interval data

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Abstract

Vector Fitting is an effective technique for rational approximation of LTI systems. It has been extended to fit the magnitude of the transfer function in absence of phase data. In this paper, magnitude Vector Fitting is modified to work on inequalities which the magnitude of the transfer function has to satisfy, instead of least squares approximation. The new interval version of the magnitude Vector Fitting is proved valuable for multiband filter design and the fitting of noisy magnitude spectra.

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1. Introduction

Vector Fitting is a robust approach for the identification of transfer functions from measured or calculated frequency responses [13,14]. Over the years, several extensions and improvements of this methodology have emerged [4,12,6,10,11,5,15]. In the case where phase data are missing, the magnitude Vector Fitting [7–9] can be applied to model only the squared magnitude of the transfer function.

Paper [17] introduces a variant of rational interpolation, which tries to fit *interval data* instead of plain function values. By interval data fitting is meant the process of finding a function $r(s)$ which satisfies inequalities

$$a_i \leq r(s_i) \leq b_i \quad (1)$$

with $i = 1, \dots, n$. For each data point s_i , an interval $[a_i, b_i]$ is provided. Paper [17] focused on the case where $r(s)$ was a rational function, while others [16] have discussed the problem in case of polynomial interpolation.

This paper adapts the magnitude Vector Fitting methodology to work on interval data. It takes the Vector Fitting approximant and constructs linear inequalities which make the approximant's value lie within certain bounds at given data points. This methodology searches for a function $\tilde{H}(s)$, such that the squared magnitude of \tilde{H} lies inside certain intervals for all data points s_i

$$a_i \leq |\tilde{H}(s_i)|^2 \leq b_i \quad (2)$$

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Section 2 will repeat the fundamentals of the magnitude Vector Fitting methodology, while Section 3 introduces some modifications to the algorithm to accommodate interval data. Sections 4 and 5 solve the systems of inequalities which arose in the other sections. Finally, some practical examples and applications are shown in Sections 6 and 7 to demonstrate the new methodology.

2. Magnitude Vector Fitting

Let the proper real-rational transfer function $H(s)$ (with $s = j\omega$) be defined by its poles p_j , zeros z_j and gain k as follows

$$H(s) = k \frac{\prod_{j=1}^M (s - z_j)}{\prod_{j=1}^M (s - p_j)} \tag{3}$$

then its squared magnitude is given by

$$|H(s)|^2 = H(s)H^*(s) = |k|^2 \frac{\prod_{j=1}^M (s - z_j)(-s - z_j^*)}{\prod_{j=1}^M (s - p_j)(-s - p_j^*)} = |k|^2 \frac{\prod_{j=1}^M (s - z_j)(s + z_j^*)}{\prod_{j=1}^M (s - p_j)(s + p_j^*)} \tag{4}$$

From the transfer function $H(s)$, being a quotient of two polynomials with real coefficients, it follows that both numerator and denominator of $|H(s)|^2 = H(s)H(-s)$ are real and even polynomials in s . Hence, if x is a pole (or zero), then $-x$, x^* and $-x^*$ are also poles (or zeros).

Following the calculations in [7,8], (4) can also be written in a modified pole/residue form, with poles and residues occurring in conjugate pairs.

$$|H(s)|^2 = \sum_{j=1}^M r_j \left(\frac{1}{s - p_j} - \frac{1}{s + p_j} \right) + d \tag{5}$$

Magnitude Vector Fitting proceeds by modeling both $|(H\sigma)(s)|^2$ and $|\sigma(s)|^2$ as defined in [13] both at the same time, using the same set of basis functions

$$f_0(s) = 1 \tag{6}$$

and

$$f_j(s) = \frac{1}{s - p_j} - \frac{1}{s + p_j} \quad j = 1, \dots, M \tag{7}$$

resulting in the following equations

$$\begin{pmatrix} |\sigma(s)|^2 \\ |(H\sigma)(s)|^2 \end{pmatrix} = \begin{pmatrix} k_0 + \sum_{j=1}^M k_j f_j(s) \\ c_0 + \sum_{j=1}^M c_j f_j(s) \end{pmatrix} \tag{8}$$

Note that this is quite similar to regular Vector Fitting [13], where one fits $\sigma(s)$ and $(H\sigma)(s)$ at the same time.

Proceeding as with the standard Vector Fitting methodology, one evaluates these equations in all data points s_k , multiplies the first equation of (8) by $|H(s_k)|^2$ and observes that the two expressions should be equal in all sample

points. This results in a least squares system in the unknowns k_j and c_j . In order to avoid the all-zero solution, one either fixes one of the coefficients at 1, or adds a “relaxation” condition as an additional matrix equation, see [12]

$$\sum_{k=1}^N \left(\sum_{j=1}^M k_j f_j(s_k) + k_0 \right) = N \tag{9}$$

The reason for this is that as the poles become close to the real poles of the system, $|\sigma(s)|^2$ approaches one, and Eq. (9) is just the sum of N ones.

The above procedure does not guarantee that the residues (r_j and r_k) belonging to complex conjugate poles (p_j and p_k) are complex conjugates, which is required when $H(s)$ is a real-rational transfer function. Complex conjugate residues can be enforced by replacing $f_j(s)$ and $f_k(s)$ by $g_j(s) := f_j(s) + f_k(s)$ and $g_k(s) := j(f_j(s) - f_k(s))$ respectively, whenever p_j and p_k are complex conjugate poles. For real poles p_l one keeps $g_l(s) = f_l(s)$. Doing so will result in a real system of linear equations instead of a complex one. Furthermore, the residues (coefficients) of the original $f_j(s)$ and $f_k(s)$ can be found as $c_j \pm jc_k$, with c_j and c_k the (real) coefficients for $g_j(s)$ and $g_k(s)$, respectively.

Next, one needs to find the zeros of $|\sigma|^2$, because they can be identified as the true poles of the fitted $H(s)$, just like in regular Vector Fitting. Once these zeros have been found, magnitude Vector Fitting proceeds in an iterative fashion by replacing the poles p_j with the new poles (i.e. the zeros of $|\sigma(s)|^2$) and continuing iteratively.

In order to find the zeros of $|\sigma|^2$ (i.e. the new poles for the next iteration), one can use the same technique as in standard Vector Fitting. Rewriting (7) as

$$f_j(s) = \frac{2p_j}{s^2 - p_j^2} \tag{10}$$

one can reformulate $|\sigma(s)|^2$ as

$$|\sigma(s)|^2 = \sum_{j=1}^M \frac{\tilde{r}_j}{s^2 - q_j} + k_0 \tag{11}$$

with $q_j = p_j^2$ and $\tilde{r}_j = 2p_j k_j$. Note that this is a plain pole/residue form in the variable s^2 . A number of methods are known for finding the zeros of a pole/residue form. In magnitude Vector Fitting, it is customary to compute the zeros as the eigenvalues of $A - BD^{-1}C^T$, where A is the diagonal matrix containing the poles, B is a column matrix of ones, C is the column matrix containing the residues \tilde{r}_j and D is equal to the scalar k_0 .

After a few iterations, the poles have settled in their final positions. What remains is to get a final approximation of $|H(s)|^2$. This step is the magnitude Vector Fitting analogy of the residue identification step in standard Vector Fitting. This approximation of $|H(s)|^2$ can be found by solving the following system of linear equations in the least squares sense

$$|H(s)|^2 = c_0 + \sum_{j=1}^M c_j f_j(s) \tag{12}$$

where the p_j are the poles found in the last iteration. Again, the residues c_j will be complex conjugate if one solves the modified system using the g_j 's instead. Discarding zeros and poles having positive real parts gives a minimum phase shift function whose magnitude matches the magnitude of $|H(s)|$, see [9].

3. Interval data

Some authors [16,17] have applied fitting techniques to interval data instead of regular data. In the Vector Fitting context, suppose one has samples at frequencies s_1, \dots, s_n with observed intervals $[\check{H}_1, \hat{H}_1], \dots, [\check{H}_n, \hat{H}_n]$. Instead of trying to fit $|H(s)|^2$ in a least squares sense, one tries to find a function H such that

$$\check{H}_k \leq |H(s_k)|^2 \leq \hat{H}_k \tag{13}$$

(the upper and lower bounds are put on the squared magnitude of $H(s_k)$).

If these bounds are not too stringent, it is possible to use optimization algorithms to find a solution for H . This section describes how magnitude Vector Fitting can be applied in the context of interval data.

When faced with interval data, one cannot proceed from Eq. (8) by multiplying $|\sigma(s)|^2$ by $|H(s)|^2$ (like with traditional Vector Fitting) as there is no exact data available. On the other hand, one can get a set of inequalities, using (13)

$$\begin{cases} |\sigma(s_k)H(s_k)|^2 \leq \hat{H}_k \left(\sum_{j=0}^M k_j g_j(s_k) \right) \\ |\sigma(s_k)H(s_k)|^2 \geq \check{H}_k \left(\sum_{j=0}^M k_j g_j(s_k) \right) \end{cases} \tag{14}$$

The left-hand side of these inequalities is still equal to the other equation from (8). Therefore, one can rewrite (14) as matrix inequalities

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{H}_k & \hat{H}_k g_1(s_k) & \dots & \hat{H}_k g_M(s_k) & -1 & -g_1(s_k) & \dots & -g_M(s_k) \\ \vdots & \vdots \\ -\check{H}_k & -\check{H}_k g_1(s_k) & \dots & -\check{H}_k g_M(s_k) & 1 & g_1(s_k) & \dots & g_M(s_k) \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_M \\ c_0 \\ c_1 \\ \vdots \\ c_M \end{pmatrix} \geq 0 \tag{15}$$

where “ ≥ 0 ” means that every component of the vector has to be greater than 0. If one writes the matrix on the left-hand side as U , and the vector of unknowns as λ , one obtains the general problem formulation $U\lambda \geq 0$. Section 4 explains how to deal with this problem.

The second problem that needs to be solved when using magnitude Vector Fitting for interval data, is that of the residue identification. In that case, one has to solve the following system of linear inequalities

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ -1 & -g_1(s_k) & \dots & -g_M(s_k) \\ \vdots & \vdots & \vdots \\ 1 & g_1(s_k) & \dots & g_M(s_k) \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{pmatrix} \geq \begin{pmatrix} -\hat{H}_1 \\ \vdots \\ -\hat{H}_N \\ \check{H}_1 \\ \vdots \\ \check{H}_N \end{pmatrix} \tag{16}$$

where the inequality means component-wise inequality. This problem is of the form $U\lambda \geq V$ and will be solved in Section 5.

4. The pole identification problem

The pole identification problem is similar to the one solved by [17], involving rational interpolation on interval data. Their technique is also applicable in the magnitude Vector Fitting context, and more specifically to the pole identification problem

$$U\lambda \geq 0 \quad (17)$$

Observe that the vectors λ satisfying this equation all lie in a *polyhedral cone*, i.e. a cone bounded by hyperplanes through the origin, namely

$$u_j\lambda \geq 0 \quad (18)$$

(with u_j the j th row of U). Furthermore, if λ is a solution, then $\alpha\lambda$ is an equivalent solution for each positive scalar α . Therefore, solutions are actually *rays*

$$\{\alpha\lambda | \alpha > 0\} \quad (19)$$

The optimal solution discussed in [17] is the ray representing the so-called Chebyshev direction. It is the ray which lies the “furthest away” from the faces of the polyhedral cone. This last problem is proven to be equivalent to finding the point λ closest to the origin, satisfying the linear inequalities $u_j\lambda \geq 0$ by a margin δ , so that

$$\frac{u_j}{\|u_j\|}\lambda \geq \delta \quad (20)$$

or

$$u_j\lambda - \delta\|u_j\| \geq 0 \quad (21)$$

The problem at hand is in fact a quadratic programming problem:

$$\operatorname{argmin}_{\lambda \in \mathbb{R}^{2M+2}} \|\lambda\|^2 \quad (22)$$

such that

$$u_j\lambda - \delta\|u_j\| \geq 0 \quad (23)$$

where the square of the norm is chosen as objective function, as it is quadratic in the components of λ . The example below uses the MATLAB `quadprog` method to solve this problem.

Alternatively, the problem can be seen as a *convex optimization problem* [3], that is, a problem of the form

$$\operatorname{argmin}_{\lambda \in \mathbb{R}^d} h_0(\lambda) \quad (24)$$

such that

$$h_j(\lambda) \geq 0 \quad (25)$$

for convex functions h_0 and h_j . Note that the squared norm $\|\lambda\|^2$ is convex, and the linear functions h_j are convex.

5. The residue identification problem

The residue identification problem is slightly different from the pole identification, as it consists of inequalities of the form

$$u_j\lambda \geq v_j \quad (26)$$

where u_j is the j th row of U , and v_j is the j th component of V . These inequalities represent (hyper) half-planes in \mathbb{R}^d . The intersection of such a set of half-planes is a multidimensional polyhedron, and all λ satisfying (26) are interior or boundary points of this polyhedron. A natural optimal solution of such a system of inequalities would be the point

furthest away from the boundaries (the so-called Chebyshev center [3]). For any point in the interior of the polyhedron, the distance to the j th boundary is given by:

$$r_j := \frac{u_j}{\|u_j\|} \lambda - \frac{v_j}{\|u_j\|} \tag{27}$$

Therefore, the optimization problem to find the Chebyshev center is identified as

$$\operatorname{argmax}_{r \geq 0} r \tag{28}$$

such that

$$u_j \lambda + -r \|u_j\| v_j \tag{29}$$

Again, two alternate solution methods are at hand. First, the problem can be rewritten as a linear programming problem

$$\operatorname{argmax}_{\mu \in \mathbb{R}^{d+1}} (1, 0, \dots, 0) \mu \tag{30}$$

such that

$$\begin{pmatrix} 1 & 0 \\ \|u_j\| & U \end{pmatrix} \mu \geq \begin{pmatrix} 0 \\ V \end{pmatrix} \tag{31}$$

with $\mu = (r \lambda^T)^T$. Second, the problem can also be treated as a convex optimization problem (see [3] Section 4.3.1). As two other alternatives, the methods of Section 4 can be used, too. Add an extra unknown ξ and solve

$$-\xi V + U \lambda \geq 0 \tag{32}$$

for λ and ξ . The implementation for the example uses the MATLAB `linprog` method to solve the linear programming problem.

6. Examples : Multiband Filter Design

In this section, it is shown that the proposed algorithm can be applied to the design of multiband microwave filters [18]. Multiband filters pass certain parts of the frequency range while filtering out other parts of the frequency range. This divides the frequency spectrum into a series of pass bands and stop bands which are separated by transition bands. For the design of such filters, several upper and lower bounds are imposed on different parts of the magnitude spectrum. The goal of the interval-based magnitude Vector Fitting method is to compute an accurate transfer function that satisfies these constraints.

As a first example, the modeling of a bandpass filter is considered that exhibits the following magnitude characteristics [2]:

$$-\infty \text{ dB} \leq |H(s)| \leq -20 \text{ dB} \quad \text{for} \quad 4.5 \text{ GHz} \leq \omega \leq 4.7 \text{ GHz} \tag{33}$$

$$0 \text{ dB} \geq |H(s)| \geq -3 \text{ dB} \quad \text{for} \quad 4.9 \text{ GHz} \leq \omega \leq 5.1 \text{ GHz} \tag{34}$$

$$-\infty \text{ dB} \leq |H(s)| \leq -20 \text{ dB} \quad \text{for} \quad 5.3 \text{ GHz} \leq \omega \leq 5.5 \text{ GHz} \tag{35}$$

It is noted that the bounds are imposed on the magnitude, rather than the squared magnitude function. The new interval-based algorithm, as described in this paper, is applied to compute a 4-pole transfer function through the intervals at 100 equidistantly spaced frequencies. The magnitude of the resulting rational approximation is shown in Fig. 1 and it is seen that the calculated transfer function satisfies the design specifications, as desired.

The second example is another bandpass filter with following characteristics:

$$-\infty \text{ dB} \leq |H(s)| \leq -13.2 \text{ dB} \quad \text{for} \quad 0 \leq \omega \leq 0.2\pi \tag{36}$$

$$0 \text{ dB} \geq |H(s)| \geq -0.5 \text{ dB} \quad \text{for} \quad 0.25\pi \leq \omega \leq 0.45\pi \tag{37}$$

$$-\infty \text{ dB} \leq |H(s)| \leq -23 \text{ dB} \quad \text{for} \quad 0.52\pi \leq \omega \leq \pi \tag{38}$$

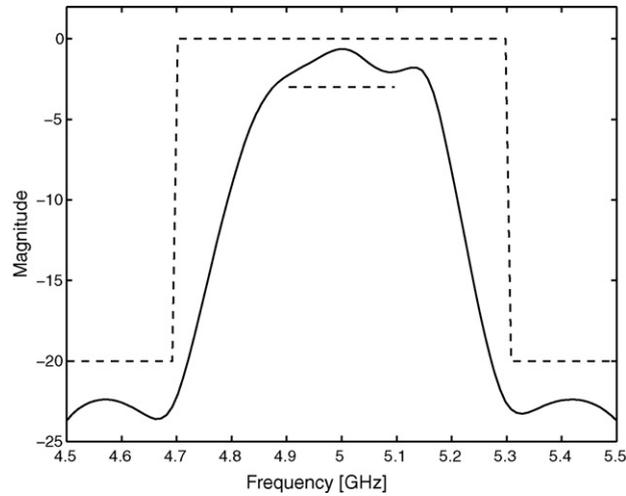


Fig. 1. Bandpass filter (example 1) : magnitude approximation using intervals.

It is based on a similar example that was reported previously in [1]. However, in this case, the pass band (37) is chosen to be twice as small. This leads to more restrictive constraints and makes the modeling somewhat more difficult. Again, the interval-based magnitude Vector Fitting is applied and a 4-pole transfer function is computed through the intervals at 100 equidistantly spaced frequencies. The magnitude of the resulting rational approximation is shown in Fig. 2 and it is seen that an excellent solution is obtained.

7. Example : Fitting of noisy magnitude spectra

It is known that the convergence properties of the original Vector Fitting algorithm become severely impaired if the response to be fitted is contaminated with noise. To relieve these difficulties, some relaxation based fitting techniques with an optional hard pole relocation procedure can be applied [11]. However, if the maximum deviation of the noise is known in advance, then this information can be used as an interval bound for the proposed algorithm. The example below illustrates that the interval-based magnitude Vector Fitting can be useful to compute transfer functions from noisy magnitude spectra.

The magnitude of the reflection coefficient S_{11} of a lowpass filter is calculated by a full-wave planar electromagnetic simulator over the frequency range from 2 GHz to 4 GHz. As an illustration, some random uniformly distributed noise

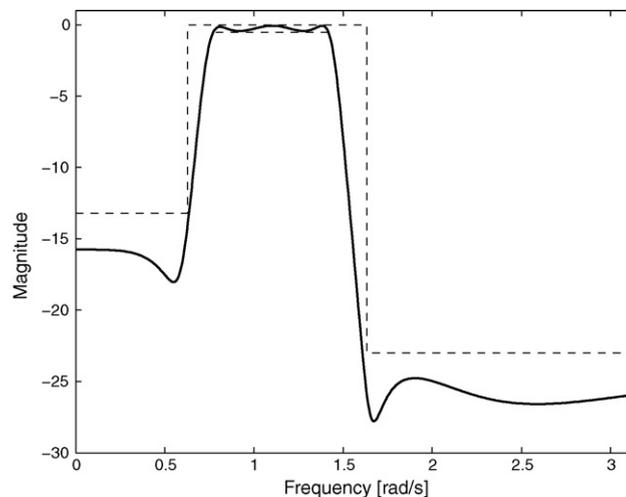


Fig. 2. Bandpass filter (example 2) : magnitude approximation using intervals.

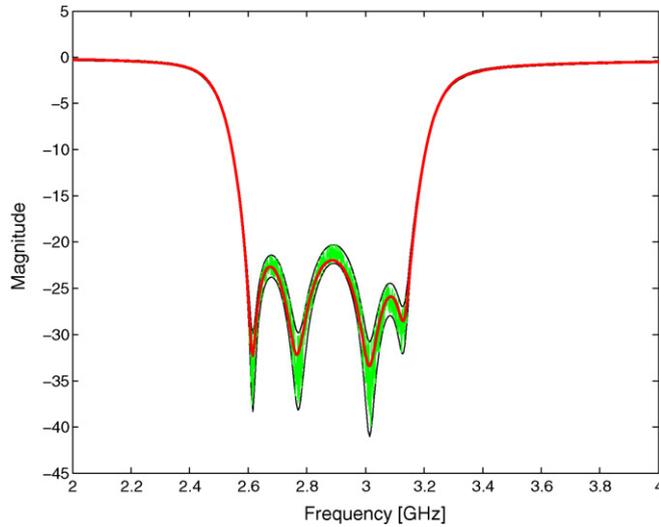


Fig. 3. Lowpass filter (noise) : magnitude approximation using intervals.

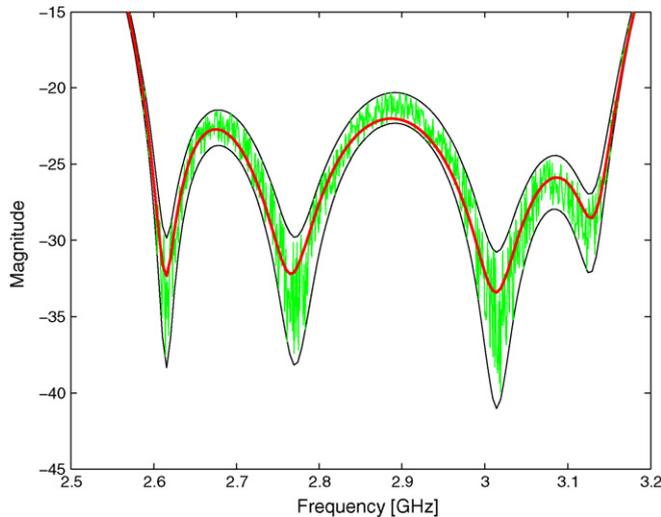


Fig. 4. Lowpass filter (noise) : zoom of Fig. 3.

with a maximum deviation of ± 0.01 is added to the data. Figs. 3 and 4 show a visualization of the noisy frequency response (green curve), and the extremal boundaries of the noise (black curves). The interval-based magnitude Vector Fitting algorithm is applied to compute a 10-pole transfer function through 300 equidistantly spaced frequencies, and the magnitude of the corresponding transfer function is shown for comparison (red curve). It can be seen that the response of the transfer function lies within the predefined interval boundaries, and a fair agreement with the noisy frequency response is observed.

8. Conclusion

A new variant of magnitude Vector Fitting has been derived, that calculates a transfer function based on interval data. Several examples show that it can be applied successfully to the design of multiband microwave filters and the fitting of noisy magnitude spectra. Interval fitting techniques are an alternative to standard Vector Fitting which is definitely worth further exploration.

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