Experimental Analysis on the Relaxation of Macromodeling Methods

Dirk Deschrijver, Marlize Schoeman, Tom Dhaene *Senior Member, IEEE*, Petrie Meyer *Member, IEEE*

***Abstract*— Relaxed Vector Fitting is known to be a reliable tool for the macromodeling of noisy frequency reponses. The strength of this method originates from the introduction of a "relaxed" non-triviality condition. In this paper, some alternative, generalized formulations are proposed and compared to the classical scheme.**

***Index Terms*— Macromodeling, Vector Fitting, Rational Approximation, System Identification.**

I. Introduction

Rational function identification from measured or simulated data becomes increasingly important for the modeling of linear systems and devices. Nowadays, the Vector Fitting (VF) method [1] has become a standard approach in the field to calculate such a transfer function in a reliable way. In the VF method, the numerator $N(s)$ and denominator $D(s)$ of the transfer function are represented as a linear combination of P partial fractions, based on a prescribed set of poles $-a_p$, such that

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{p=1}^{P} c_p/(s+a_p)}{\tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p/(s+a_p)} \tag{1}$$

with $s = j2\pi f$. The denominator has an additional basis function which equals the constant value 1, and the coefficients c_p and $\tilde{c}_p$ represent the model coefficients. Given a set of Laplace data samples $(s_k, H(s_k))$, the transfer function should match the data in a least-squares (LS) sense, such that $R(s_k) \simeq H(s_k)$, for $k = 0, ..., K$. Some further improvements in terms of conditioning can be made by using a set of orthonormal rational functions, leading to the Orthonormal Vector Fitting (OVF) method [2].

The numerator and denominator of (1) can be factorized as follows

$$N(s) = \sum_{p=1}^{P} \frac{c_p}{s+a_p} = \frac{\prod_{p=1}^{P-1}(s+z_{p,n})}{\prod_{p=1}^{P}(s+a_p)} \tag{2}$$

$$D(s) = \tilde{c}_0 + \sum_{p=1}^{P} \frac{\tilde{c}_p}{s+a_p} = \frac{\prod_{p=1}^{P}(s+z_{p,d})}{\prod_{p=1}^{P}(s+a_p)} \tag{3}$$

and the transfer function $R(s)$ is easily obtained as

$$R(s) = \frac{N(s)}{D(s)} = \frac{\prod_{p=1}^{P-1}(s+z_{p,n})}{\prod_{p=1}^{P}(s+z_{p,d})} = \sum_{p=1}^{P} \frac{\alpha_p}{s+z_{p,d}}. \tag{4}$$

Dirk Deschrijver and Tom Dhaene are with the Ghent University, and the University of Antwerp, Belgium. Marlize Schoeman and Petrie Meyer are with the University of Stellenbosch, South Africa.

This work was supported by the Fund for Scientific Research Flanders (FWO Vlaanderen) and the Science and Innovation Administration Flanders (BSTC project).

The poles $z_d = \{-z_{1,d}, ..., -z_{P,d}\}$ can be calculated directly as the zeros of the minimal state-space realization of $D(s)$, so the calculation of the α values reduces to a linear problem. In order to relocate the poles to a better position, a Sanathanan-Koerner (SK) iteration [3] can be applied, using an implicit weighting scheme. This means that the coefficients $d^{(t)}$ and $\tilde{d}^{(t)}$ of the weighted numerator $(N^{(t)}/D^{(t-1)})$ and denominator $(D^{(t)}/D^{(t-1)})$ are estimated, as shown in equation (7), rather than the coefficients $c^{(t)}$ and $\tilde{c}^{(t)}$ of the numerator $(N^{(t)})$ and denominator $(D^{(t)})$ themselves, as shown in equation (6) with $w_k = 1/D^{(t-1)}(s_k)$. This fact does not pose a problem, as the introduction of this weighting does not influence the zeros of $D^{(t)}$. The implicit scheme, however, is numerically more reliable if the poles are not optimally chosen. The reader is referred to [2], [4] for more details about this procedure.

Experience with the original VF algorithm has shown that its convergence properties become severely impaired if the response to be fitted is contaminated with noise [5]. It was shown in [6] that this problem is related to the adopted LS normalization where $\tilde{d}_0^{(t)}$ is set equal to 1. In [6], a modification to the (O)VF algorithm was introduced that alleviates these difficulties by improving the normalization of the transfer function coefficients and the linearization of the SK-iteration at the same time. As the iteration converges, it is assumed that $D^{(t-1)}(s_k)$ will approach $D^{(t)}(s_k)$, so an unbiased fitting would be achieved if $D^{(t)}(s_k)/D^{(t-1)}(s_k)$ approaches unity at all frequencies. In order to obtain this goal, a more relaxed non-triviality condition is added as an additional row in the system matrix

$$\Re e \left\{ \sum_{k=0}^{K} \left(\tilde{d}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{d}_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} \right) \right\} = K+1. \tag{8}$$

This equation is given a LS weighting in relation to the size of H

$$weight = \|H(s)\| / (K+1). \tag{9}$$

It was shown that this approach significantly improves the relocation of poles if the system equations are overdetermined, or when the data is corrupted with noise.

II. Alternative relaxation constraints

In this section, some alternative constraints are proposed. As the fitting accuracy of RVF models often converges quite well, it will be investigated if these alternative formulations can speed up the convergence process of the algorithm. It is noted that the following propositions are merely a selection of many other possible relaxation constraints.

$$\arg\min \left(\sum_{k=0}^{K} \left| \frac{N^{(t)}(s_k)}{D^{(t-1)}(s_k)} - \frac{D^{(t)}(s_k)}{D^{(t-1)}(s_k)} H(s_k) \right|^2 \right) \tag{5}$$

$$= \arg\min_{c^{(t)}, \tilde{c}^{(t)}} \left(\sum_{k=0}^{K} \left| w_k \left[\sum_{p=1}^{P} \frac{c_p^{(t)}}{s_k + a_p} - \left(\tilde{c}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{c}_p^{(t)}}{s_k + a_p} \right) H(s_k) \right] \right|^2 \right) \tag{6}$$

$$= \arg\min_{d^{(t)}, \tilde{d}^{(t)}} \left(\sum_{k=0}^{K} \left| \sum_{p=1}^{P} \frac{d_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} - \left(\tilde{d}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{d}_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} \right) H(s_k) \right|^2 \right). \tag{7}$$

A. Proposition 1

a) The condition (8) requires that $D^{(t)}(s)/D^{(t-1)}(s)$ approaches unity, which implies convergence of the algorithm. Similarly, one can propose an analogous reasoning for $N^{(t)}(s)/N^{(t-1)}(s)$. It is easy to see that

$$\begin{aligned} \frac{N^{(t)}(s)}{N^{(t-1)}(s)} &= \frac{N^{(t)}(s)}{D^{(t-1)}(s)} \frac{D^{(t-1)}(s)}{N^{(t-1)}(s)} \\ &= \left(\sum_{p=1}^{P} \frac{d_p^{(t)}}{s + z_{p,d}^{(t-1)}} \right) \left(R_{fit}^{(t-1)}(s) \right)^{-1}, \end{aligned} \tag{10}$$

where $R_{fit}^{(t-1)}(s)$ represents the data samples obtained from the fitting model, which was constructed in iteration $t-1$. Therefore, equation (8) can be generalized to equation (11). It is observed that this relaxation constraint makes use of both the coefficients $d_p^{(t)}$ and $\tilde{d}_p^{(t)}$, which exploits the sparsity of the classical relaxation constraint (8). The weighting of this equation is updated accordingly: $weight = \|H(s)\| / (2K + 2)$.

b) Similarly, one can replace the values of $R_{fit}^{(t-1)}(s)$ by $H(s)$ in equation (11), since the data which comes from the fitting model should converge towards the actual frequency response.

B. Proposition 2

If $D^{(t)}(s)/D^{(t-1)}(s)$ approaches the constant function unity, then it follows directly that the v^{th} order frequency derivatives ($v > 0$) of this function go to zero. Provided that these higher-order derivatives are defined as

$$\frac{\partial^v}{\partial s^v} \left(\frac{D^{(t)}(s)}{D^{(t-1)}(s)} \right) = \sum_{p=1}^{P} (-1)^v v! \tilde{d}_p^{(t)} \left(s + z_{p,d}^{(t-1)} \right)^{-(v+1)}, \tag{12}$$

equation (8) can be generalized to equation (13). for an arbitrary positive value of V. Note that (13) reduces to (8) if $v = 0$.

C. Proposition 3

Since $D^{(t)}(s)/D^{(t-1)}(s)$ approaches the constant function unity, one can assume that the imaginary part of this function eventually goes to zero. Therefore, equation (8) can be generalized to equation (14).

D. Proposition 4

The relaxation constraint imposes that $D^{(t)}(s)/D^{(t-1)}(s)$ approaches unity at the frequencies $s_k = j\omega_k$, for $k = 0, ..., K$. One can equally well impose that the function approaches unity at the complex frequencies $s_k = \sigma_k + j\omega_k$, for arbitrary positive values of σ_k. In this paper, $\sigma_k^{(\gamma)} = \gamma\rho\omega_K$ and $\rho = 0.01$, hence a generalized constraint is obtained as shown in equation (15). Clearly, the introduction of $\sigma_k^{(\gamma)}$ results in the shifting of the poles. Note also that (15) reduces to (8) if $\rho = 0$. The weighting of this equation is updated accordingly: $weight = \|H(s)\| / (2K + 2)$.

III. Relaxed SK Weighting

In this section, an alternative way of relaxation is proposed, which is based on an explicit weighting of the SK-iteration. Recall that the use of an explicit weighting corresponds to solving the coefficients $c_p^{(t)}$ and $\tilde{c}_p^{(t)}$ of $N(s)$ and $D(s)$, provided that each equation is given a weighting as shown in equation (6), with $w_k = 1/D^{(t-1)}(s_k)$. In [7], 't Mannetje proposed to relax this weighting in each iteration, by raising it to the power r such that w_k in (6) is generalized as follows

$$w_k = \left(\frac{1}{D^{(t-1)}(s_k)} \right)^r = \left(\frac{\prod_{p=1}^{P} (s_k + a_p)}{\prod_{p=1}^{P} (s_k + z_{p,d}^{(t-1)})} \right)^r. \tag{16}$$

Clearly, $-z_{p,d}^{(t-1)}$ represent the zeros of $D^{(t-1)}(s_k)$, which are also equivalent to the poles of $D^{(t)}(s_k)/D^{(t-1)}(s_k)$. Note that (6) reduces to Levi's estimator [8] if $r = 0$, and to the classical SK-iteration if $r = 1$. In each iteration of the algorithm, the optimal choice of r can be determined, by using standard optimization techniques. According to our practical experiments, the optimal value of r is usually located in the interval $0 < r < 2$. It was shown in [7] that this approach often improves the convergence properties of the explicitly-weighted SK iteration, particularly when the data is contaminated with noise.

IV. Example

The reflection coefficient S_{11} of an RDRAM channel with 16 memory devices was simulated, and approximated from DC up to 2.5 GHz by a strictly proper transfer function. The data shows large reflection, as can be seen from the magnitude response in Figure 1. This frequency domain data is used to compare the VF and RVF approaches to the alternative

$$\Re e\left\{\sum_{k=0}^{K}\left(\sum_{p=1}^{P}\left[\left(\frac{d_p^{(t)}}{s_k+z_{p,d}^{(t-1)}}\right)\left(R_{fit}^{(t-1)}(s_k)\right)^{-1}\right]+\tilde{d}_0^{(t)}+\sum_{p=1}^{P}\frac{\tilde{d}_p^{(t)}}{s_k+z_{p,d}^{(t-1)}}\right)\right\}=2K+2. \quad (11)$$

$$\Re e\left\{\sum_{k=0}^{K}\left(\tilde{d}_0^{(t)}+\sum_{p=1}^{P}\sum_{v=0}^{V}(-1)^v v!\tilde{d}_p^{(t)}\left(s_k+z_{p,d}^{(t-1)}\right)^{-(v+1)}\right)\right\}=K+1. \quad (13)$$

$$\Re e\left\{\sum_{k=0}^{K}\left(\tilde{d}_0^{(t)}+\sum_{p=1}^{P}\frac{\tilde{d}_p^{(t)}}{s_k+z_{p,d}^{(t-1)}}\right)\right\}+j\Im m\left\{\sum_{k=0}^{K}\left(\tilde{d}_0^{(t)}+\sum_{p=1}^{P}\frac{\tilde{d}_p^{(t)}}{s_k+z_{p,d}^{(t-1)}}\right)\right\}=K+1. \quad (14)$$

$$\Re e\left\{\sum_{\gamma=0}^{1}\sum_{k=0}^{K}\left(\tilde{d}_0^{(t)}+\sum_{p=1}^{P}\frac{\tilde{d}_p^{(t)}}{j\omega_k+\left(\sigma_k^{(\gamma)}+z_{p,d}^{(t-1)}\right)}\right)\right\}=2K+2. \quad (15)$$

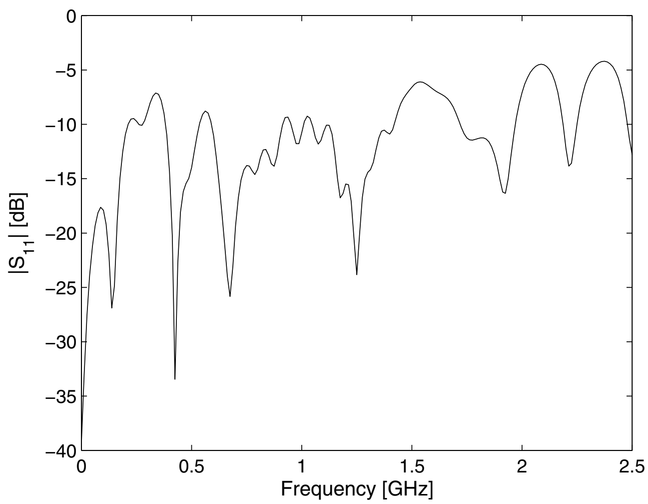

Fig. 1. Magnitude of the data (S_{11}).

techniques presented in this paper. The starting poles are chosen as a prescribed set of complex conjugate pairs on a straight line with small real parts ($-0.01\omega_{\max}$), and their imaginary parts are equidistantly spread over the frequency range of interest. At DC, the complex conjugate pair of poles with zero imaginary part is replaced by a single real pole in order to avoid a singular set of equations. In this example, all frequencies are scaled by 10^9. Unstable poles are flipped into the left-half plane during each iteration, in order to enforce stability of the poles.

First, the alternative relaxation constraints as described in Section II are used to calculate a 39-pole transfer function. Figures 2 and 3 show the evolution of the RMS error in terms of iteration count when no additional noise, or 1% additional noise is added to the data. Clearly, the influence of the relaxation gives an improvement compared to the classical VF method, however, there is no significant difference between these different flavours when they are compared in terms of convergence speed and accuracy. This observation indicates that the improvement of RVF over VF results mainly from the fact that the constant term of the denominator is “free”, rather than the actual choice of the relaxation constraint as presented in this paper. In some examples, one method can give some benefit compared to the others, but this improvement is not always consistent. Especially the behaviour of Proposition 3 was found to be highly unreliable at some instances.

Usually, ’t Mannetje (see Section III) gives a result, lying somewhere inbetween VF and RVF. When considering iteration 11 of Figure 2, it can be seen that ’t Mannetje is less accurate than VF. This interesting fact shows that the optimal value of r in a given iteration, does not necessarily imply that better results will be obtained in successive iterations. This indicates that a better result may occur if a sub-optimal choice is made at some point.

If the number of poles is increased to 43, it is found that ’t Mannetje outperforms VF and RVF, as shown in Figure 4. This indicates that the convergence of these relaxation-based methods, like RVF (see Section II), is not guaranteed to be optimal. It is also noted that the final model accuracy of RVF is less accurate when 43 poles are used (Figure 4), as compared to the situation when 39 poles are used (Figure 2). This confirms the statement that RVF is not always optimal.

Figure 5 shows the RMS error of ’t Mannetje in each iteration, corresponding to the 43-pole transfer function, for various choices of $0 < r < 2$. It can be seen that the behaviour of the curve is smooth, which indicates that the optimization problem is relatively easy. Note that the RMS error of the optimal choice of r corresponds to the RMS errors shown in Figure 4. It is also observed that this error drops in successive iterations, and that it leads to better intermediate results as compared to the classical SK-iteration ($r = 1$). It should however be noted that the convergence of ’t Mannetje can sometimes be impaired by ill-conditioning, caused by the explicit weighting. Such ill-conditioning may occur e.g. if the data is poorly observable, or when the initial poles are poorly specified. The reader is referred to [4] for more details.

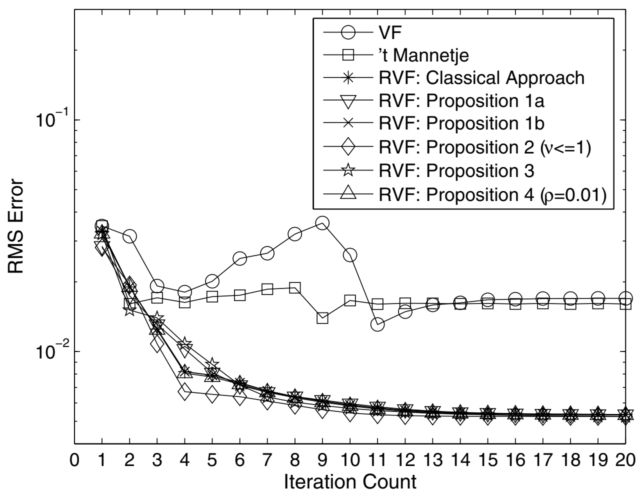

Fig. 2. RMS error vs. Iteration count (39 poles, 0% noise).

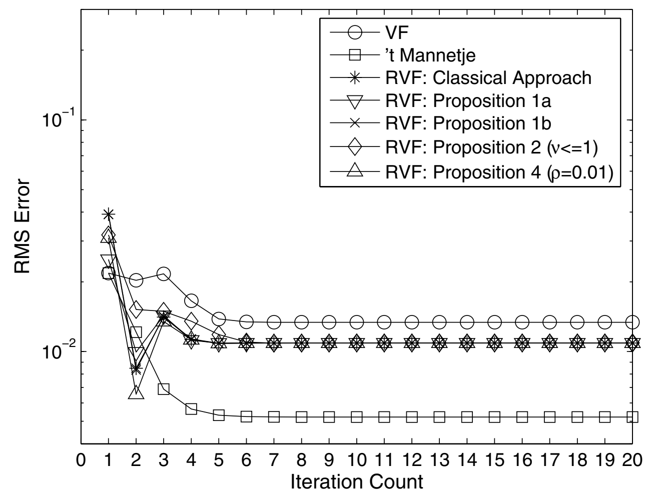

Fig. 4. RMS error vs. Iteration count (43 poles, 0% noise).

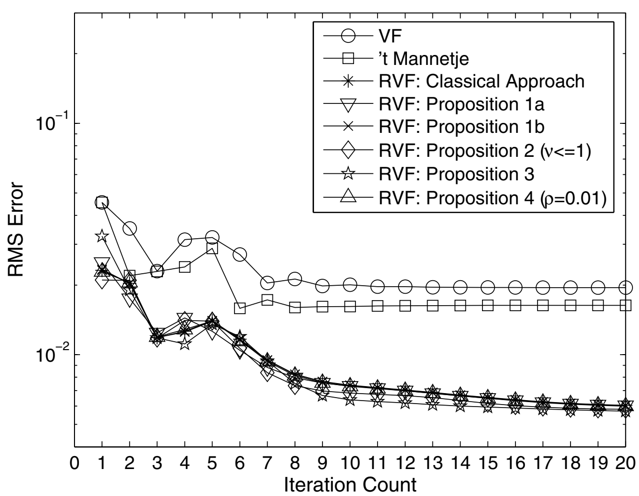

Fig. 3. RMS error vs. Iteration count (39 poles, 1% noise).

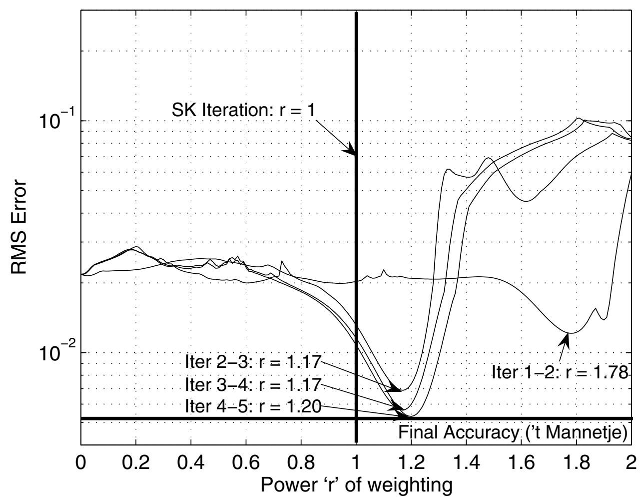

Fig. 5. Optimization of weighting for iterations 1 to 5.

V. Conclusion

It is known that the convergence of Vector Fitting can be impaired if the data is contaminated with noise. Several alternative approaches are proposed and compared on a simulation-based example. It is shown that the VF method can be improved by introducing a relaxation constraint, like e.g. the RVF method. Some alternative relaxation-based approaches are proposed, but they usually lead to models with a comparable accuracy. A comparison with the technique of 't Mannetje (which is based on an explicit weighting scheme), shows that neither approach guarantees convergence to an optimal result.

Acknowledgment

This work was supported by the Fund for Scientific Research Flanders (FWO Vlaanderen) and the Science and Innovation Administration Flanders (BSTC project).

References

[1] B. Gustavsen, and A. Semlyen, "Rational Approximation of Frequency Domain Responses by Vector Fitting", *IEEE Trans. of Power Delivery*, vol. 14, pp. 1052–1061, 1999.

[2] D. Deschrijver and T. Dhaene, "Rational Modeling of Spectral Data using Orthonormal Vector Fitting", *9th IEEE Workshop on Signal Propagation on Interconnects*, 111–114, May 2006.

[3] C. Sanathanan, and J. Koerner, "Transfer Function Synthesis as a Ratio of two Complex Polynomials", *IEEE Trans. on Automatic Control*, vol. 8, no. 1, pp. 56–58, 1963.

[4] D. Deschrijver, B. Gustavsen and T. Dhaene, "Advancements in Iterative Methods for Rational Approximation in the Frequency Domain", *IEEE Trans. of Power Delivery*, to be published, 2007.

[5] S. Grivet-Talocia, M. Bandinu, "Improving the convergence of Vector Fitting for Equivalent Circuit Extraction from Noisy Frequency Responses", *IEEE Trans. on Electromagnetic Compatibility*, 48 (1), 104–120, 2006.

[6] B. Gustavsen, "Improving the Pole Relocating Properties of Vector Fitting", *IEEE Trans. of Power Delivery*, 21 (3), 1587–1592, 2006.

[7] J.J. 't Mannetje, "Transfer Function Identification using a Complex Curve-Fitting Technique", *Journal of Mechanical Engineering Science*, 15 (5), 339–345, 1973.

[8] E.C. Levi, "Complex Curve Fitting", *IEEE Trans. on Automatic Control*, AC-4, pp. 37–43, 1959.

Copyright Information