

A comparative study of Vector Fitting and Orthonormal Vector Fitting Techniques for EMC Applications

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Abstract—Broadband rational approximations of multiport systems are of great importance for accurate transient analysis of electrical systems. To this aim, fitting techniques have been found extremely useful in providing rational models for sampled data in the frequency domain. This paper presents a comparative study of two fitting methods, namely the standard Vector fitting (VF) and the orthonormal vector fitting (OVF) techniques. The latter is found to be better conditioned, reduces the numerical sensitivity to the choice of starting poles and limits the the number of iterations and, thus, the global cpu-time to obtain the rational approximation.

I. INTRODUCTION

Broadband modeling of electrical systems, based on measurements or simulations, is often carried out by means of rational compact macromodels which allow efficient time domain simulations via convolutions or equivalent circuits [1]. In 1999 an iterative least-squares approach, called Vector Fitting (VF) [2], has been proposed and then improved in [3], [4]; over the years it has become one of the most popular techniques for rational approximation of frequency data. Since then it has been used in a number of different fields ranging from power systems [5] to electromagnetic fields [6] to printed circuit board modeling [7]. More recently an orthonormal vector fitting (OVF) technique has been proposed [8] which leads to better conditioned equations, reduces the numerical sensitivity of the model parametrization to the choice of starting poles, limits the number of required iterations and thus, the overall macromodelling time. Aim of this paper is to carry out a comparative study of VF and OVF in electromagnetic compatibility applications. The comparative study shows that OVF outperforms VF in

extracting macromodels from frequency responses as regard as the RMS error over a wide frequency range.

II. VECTOR FITTING TECHNIQUE

The major goal of the identification process, is to approximate the impedance data $(s_w, Z_{L,mn}^{dl}(s_w))$ or $Z_{C,mn}^{dl}(s_w)$, for $w = 0, \dots, W$, by an analytic transfer function. Such a rational function is defined as a ratio of 2 arbitrary polynomials

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=1}^N Res_k \varphi_k(s)}{\sum_{k=1}^D \widetilde{Res}_k \varphi_k(s)} \quad (1)$$

For continuous-time LTI systems in the frequency domain, the identification problem reduces to approximating the system parameters of numerator (Res_k) and denominator (\widetilde{Res}_k) of the transfer function $R(s)$ in a reliable and computationally efficient way. It is well known that the identification problem is numerically hard to solve, especially for broadband frequency domain responses of highly dynamical systems.

The numerical ill-conditioning is mainly caused by the use of a poorly chosen set of basis functions $\varphi_k(s)$. Typically, one uses a monomial basis, like the power series $\{1, s, s^2, \dots, s^k\}$, to expand the numerator and denominator of the transfer function. Then the associated system of equations has a Vandermonde-structure, which is notoriously ill-conditioned. Some relief can be obtained by an appropriate frequency scaling [9] or the use of alternative polynomial bases, however they don't tackle the numerical issues entirely.

In 1999, the Vector Fitting (VF) technique [2] was introduced which decomposes numerator and denominator

of the transfer function

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=1}^{N_p} Res_k \varphi_k(s) + d + se}{\sum_{k=1}^{N_p} \widetilde{Res}_k \varphi_k(s) + \widetilde{Res}_0} \quad (2)$$

into a set of N_p partial fractions

$$\varphi_k(s) = \frac{1}{(s - p_k)} \quad (3)$$

which share a common set of poles p_k . The denominator has an additional basis function which equals the constant value 1. An optional constant (d) and an optional linear term (se) can be added to the numerator expression, in order to make the transfer function $R(s)$ a proper or an improper rational function respectively.

It was shown that this approach can significantly improve the conditioning of the system equations if the starting poles are properly chosen. Also, rational basis functions have a lot of advantages compared to an arbitrary polynomial basis, like e.g. the simplification of the enforcement of system stability. The choice of this basis is combined with the use of a Sanathanan-Koerner (SK) iteration to relieve the bias of the estimator. The iteration starts from an initial guess of poles and relocates them to obtain an optimal fit. Afterwards, the residues are calculated to minimize the global fitting error.

III. ORTHONORMAL VECTOR FITTING TECHNIQUE

It has been observed that the numerical conditioning of the system equations, the accuracy of the fitting model, and the number of required iterations is highly dependent on the initial choice of starting poles. If the initial poles are chosen complex conjugate as proposed in [2], the VF method gives reliable results in a small amount of iterations.

In [8], the SK-iteration was combined with a set of orthonormal rational functions [10], which are obtained by applying a Gram-Schmidt orthonormalization on the set of partial fractions. The use of this basis leads to better conditioned equations, especially if the real part of the starting poles is non-negligible. This approach, called Orthonormal Vector Fitting (OVF), reduces the numerical sensitivity of the model parameterization to the choice of starting poles significantly, limits the number of required iterations, and reduces the overall macromodeling time [11].

A. Choice of basis functions

Both the numerator and denominator are expanded in a common set of rational basis functions $\phi_k(s)$, which

are orthonormal

$$\langle \phi_x(s), \phi_y(s) \rangle = \delta_{xy} \quad (4)$$

with respect to the continuous inner product ($1 < x, y < N_p$)

$$\langle \phi_x(s), \phi_y(s) \rangle = \frac{1}{2\pi i} \int_{i\mathbb{R}} \phi_x(s) \phi_y^*(s) ds \quad (5)$$

The orthonormal basis can easily be calculated, by applying a Gram-Schmidt orthonormalization on a set of partial fractions $\varphi_k(s)$

$$\varphi_k(s) = \frac{1}{(s - p_k)} \quad (6)$$

which leads to the following closed-form expression

$$\phi_k(s) = \frac{Q_k(s)}{\prod_{j=1}^k (s - p_j)} \quad (7)$$

$$= \frac{\kappa_k \sqrt{-2\Re(p_k)}}{(s - p_k)} \left(\prod_{j=1}^{k-1} \frac{s + p_j^*}{s - p_j} \right) \quad (8)$$

κ_k represents an arbitrary unimodular complex number, and is fixed to 1 in practice.

It is clear that these orthonormal basis functions $\phi_k(s)$ are in fact a linear combination of the partial fractions $\varphi_k(s)$ for $k = 1, \dots, N_p$, and consequently span the same space. The numerator polynomial $Q_k(s)$ is an arbitrary polynomial of order $k - 1$, and its coefficients can be obtained directly from the Gram-Schmidt orthonormalization process. Since this orthonormalization is performed analytically rather than numerically, no additional computational cost is introduced.

This basis originates from the discrete-time Takenaka-Malmquist basis [12] [13], and has later been transformed to the continuous time domain. It is a generalization of the Laguerre basis [14], where all poles $\{p_k\}$ are the same real number, and the 2-parameter Kautz basis [15] where all poles $\{p_k, p_{k+1}\}$ are the same complex conjugate pair with $p_k^* = p_{k+1}$. These bases are well-studied in literature, and the interested reader is referred to [16] for a thorough theoretical analysis.

Remark however that these basis functions (8) are not real-valued by construction. To avoid imaginary terms in the time-domain, it is desired that the polynomial coefficients of the numerator and denominator are real, such that $R^*(s) = R(s^*)$. The fitting algorithm ensures that this property is satisfied, by making the basis functions real-valued and their corresponding coefficients real. If the basisfunctions $\phi_k(s)$ and $\phi_{k+1}(s)$ correspond to a

complex conjugate pair of poles $p_k^* = p_{k+1}$, a linear combination is formed which can be made real-valued and orthonormal by applying the orthonormalization constraints. This way, the following two expressions are obtained which replace the former basis functions

$$\phi_k(s) = \frac{\sqrt{-2\Re(p_k)}(s - |p_k|)}{(s - p_k)(s - p_{k+1})} \left(\prod_{j=1}^{k-1} \frac{s + p_j^*}{s - p_j} \right) \quad (9)$$

$$\phi_{k+1}(s) = \frac{\sqrt{-2\Re(p_k)}(s + |p_k|)}{(s - p_k)(s - p_{k+1})} \left(\prod_{j=1}^{k-1} \frac{s + p_j^*}{s - p_j} \right) \quad (10)$$

In the following part of this section, it will be assumed that $\phi_k(s)$ represents (8) if p_k is real, and that $\phi_k(s)$ and $\phi_{k+1}(s)$ represent (9) and (10) respectively if p_k and p_{k+1} form a complex conjugate pair.

B. Pole identification and relocation

The goal of the identification problem is now to find the optimal values of the indeterminates $v = \{Res_k, \widetilde{Res}_k, d, e\}$ and p_k , such that the following non-linear cost function is minimized in a fast and computationally efficient way

$$\arg \min_{v,p} \sum_{w=0}^W \left| \frac{N(s_w)}{D(s_w)} - Z(s_w) \right|^2 \quad (11)$$

Based on a set of prescribed poles p_k , the remaining unknowns v can be estimated by minimizing Levi's cost function [17]. To avoid the trivial solution, one coefficient (e.g. Res_0) can be fixed to unity without loss of generality, since both numerator and denominator can be divided by the same constant value.

$$\arg \min_v \sum_{w=0}^W |N(s_w) - D(s_w)Z(s_w)|^2 \quad (12)$$

with

$$N(s_w) = \sum_{k=1}^{N_p} Res_k \phi_k(s_w) + d + s_w e \quad (13)$$

$$D(s_w) = \sum_{k=1}^{N_p} \widetilde{Res}_k \phi_k(s_w) + 1 \quad (14)$$

This optimization problem reduces to solving the following set of overdetermined equations $Ax = b$ in a

least-squares sense

$$A = \begin{pmatrix} \phi_1(s_0) & \dots & \phi_{N_p}(s_0) & 1 & s_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_1(s_W) & \dots & \phi_{N_p}(s_W) & 1 & s_W & \dots \\ -Z(s_0)\phi_1(s_0) & \dots & -Z(s_0)\phi_{N_p}(s_0) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -Z(s_W)\phi_1(s_W) & \dots & -Z(s_W)\phi_{N_p}(s_W) & \dots & \dots & \dots \end{pmatrix} \quad (15)$$

$$x = \begin{pmatrix} Res_1 & \dots & Res_{N_p} & d & e & \dots \\ \dots & \widetilde{Res}_1 & \dots & \widetilde{Res}_{N_p} & \dots & \dots \end{pmatrix}^T \quad (16)$$

$$b = \begin{pmatrix} Z(s_0) & \dots & Z(s_W) \end{pmatrix}^T \quad (17)$$

To ensure that the entries of the parameter vector are real, each equation is split in its real and imaginary part, such that

$$A = \begin{bmatrix} \Re(A) \\ \Im(A) \end{bmatrix}, \quad b = \begin{bmatrix} \Re(b) \\ \Im(b) \end{bmatrix} \quad (18)$$

Some additional improvements can be obtained by scaling the columns of A to unity norm. After simplification of (2), it becomes obvious that the poles of this function are basically the zeros of the denominator. These poles can easily be extracted by calculating the zeros of $D(s)$

$$D(s) = \sum_{k=1}^{N_p} \widetilde{Res}_k \phi_k(s) + 1 \quad (19)$$

First, its minimal state space realization (A,B,C,D) can accurately be constructed by cascading a number of smaller, first or second order filters [18]. Then, the new poles are calculated in a straightforward way by solving an eigenvalue problem

$$p_k = \text{eig}(A - BC) \quad (20)$$

Based on the identified poles (iteration step $t=0$), a Sanathanan-Koerner (SK) iteration [19] can be applied to relocate them iteratively for $t=1, \dots, T$

$$\arg \min_v \sum_{w=0}^W \left| \frac{1}{D^{(t-1)}(s_w)} \left(N^{(t)}(s_w) - D^{(t)}(s_w)Z(s_w) \right) \right|^2 \quad (21)$$

until the SK-cost function is minimized. This process is called "Pole Relocation". In practice, this means that the procedure is repeated iteratively after replacing the prescribed poles (or the poles from the previous iteration step) by the newly identified poles. Convergence occurs relatively fast if the initial set of poles is well-chosen, e.g. as was suggested in [2]

$$p_k = -\alpha + \beta i, p_{k+1} = -\alpha - \beta i \quad (22)$$

$$\alpha = \beta/100 \quad (23)$$

with imaginary parts β covering the frequency range of interest. This construction appears to give best results for the classical VF algorithm, as well as OVF. Occasionally some unstable poles may occur, which can be resolved by flipping them into the left half of the complex plane. Note that the SK iteration is not equivalent to the non-linear cost function, and that convergence to the true fundamental solution of the least-squares problem is not necessarily guaranteed. In practice however, the solutions are often sufficiently accurate for high signal-to-noise ratios and sufficiently small modelling errors. If not, they can often provide good starting points for non-linear optimization techniques.

C. Residue Identification

Once the poles are estimated, the residues Res_k , d and e of the transfer function

$$R(s) = \sum_{k=1}^{N_p} Res_k \phi_k(s) + d + se \quad (24)$$

can be identified by minimizing the linear cost function

$$\arg \min_{Res, d, e} \sum_{w=0}^W \left| \sum_{k=1}^{N_p} Res_k \phi_k(s_w) + d + s_w e - Z(s_w) \right|^2 \quad (25)$$

Similarly, this problem reduces to solving the following set of overdetermined equations $Ax = b$ in a least-squares sense

$$A = \begin{pmatrix} \phi_1(s_0) & \dots & \phi_{N_p}(s_0) & 1 & s_0 \\ \dots & \dots & \dots & \dots & \dots \\ \phi_1(s_W) & \dots & \phi_{N_p}(s_W) & 1 & s_W \end{pmatrix} \quad (26)$$

$$x = (Res_1 \dots Res_{N_p} \ d \ e)^T \quad (27)$$

$$b = (Z(s_0) \dots Z(s_W))^T \quad (28)$$

Again, each equation is split in its real and imaginary part to ensure that the parameter vector x is real. The transfer function $R(s)$ can then accurately be converted to a real minimal state space realization. If unstable poles are allowed for the fitting model, or if the transfer function is desired in Foster's Canonical form, one can resort to the partial fraction basis for the residue identification. Such representation can then easily be realized as a compact equivalent RLGC circuit.

D. Multiport systems

The extension of OVF for multi-port systems can be done entirely analogous to the matrix version of the classical VF algorithm. The basic idea is that all

elements of the system matrix are stacked in one column, and are fitted using a common set of poles. If $v = \{Res_{mn}, d_{mn}, e_{mn}, Res\}$, then equation (21) can be generalized to solving the following iterative problem

$$\arg \min_v \left(\sum_{w=0}^W \sum_{m=1}^P \sum_{n=1}^P \left| \frac{N^{(t)}(s_w) - D^{(t)}(s_w) Z_{mn}(s_w)}{D^{(t-1)}(s_w)} \right|^2 \right) \quad (29)$$

given a $P \times P$ system matrix. For more implementation details, the interested reader is referred to [2].

IV. NUMERICAL TESTS

A. Broadband PEEC modeling

The first test is carried out in the framework of the Partial Element Equivalent Circuit (PEEC) method [20]. Recently broadband PEEC modeling has drawn a great concern due to the accurate computation and implementation of partial elements, namely partial inductances and coefficients of potential [21], [22], in wide frequency range to ensure stability of the resulting model. The mutual inductive impedance between two volumes m and n is computed. Fig. 1 shows the geometry of the volumes considered. The mutual impedance $Z_{L,mn}$ has been computed numerically by using a Gaussian quadrature scheme as [23]:

$$Z_{L,mn}(s) = \frac{s\mu}{4\pi a_m a_n} \int_{v_m} \int_{v_n} \frac{e^{-s|\mathbf{r}_m - \mathbf{r}_n|/c_0}}{|\mathbf{r}_m - \mathbf{r}_n|} dv_m dv_n \quad (30)$$

Then, the impedance $Z_{L,mn}(j\omega)$ has been fitted by using

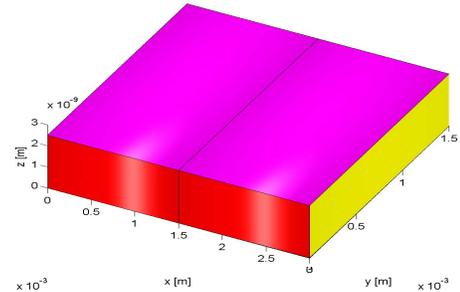


Fig. 1. Two touching volumes.

both the VF and OVF techniques in the frequency range 0.1 MHz-400 GHz, based on 50 equidistant support samples. The initial set of starting poles are chosen to be complex conjugate pairs with imaginary parts covering the frequency range of interest, as defined in equation

(22)-(23). In [2], it was argued that this heuristical scheme gives the best overall results, so this set of prescribed poles was used to do the calculations using the VF and OVF method.

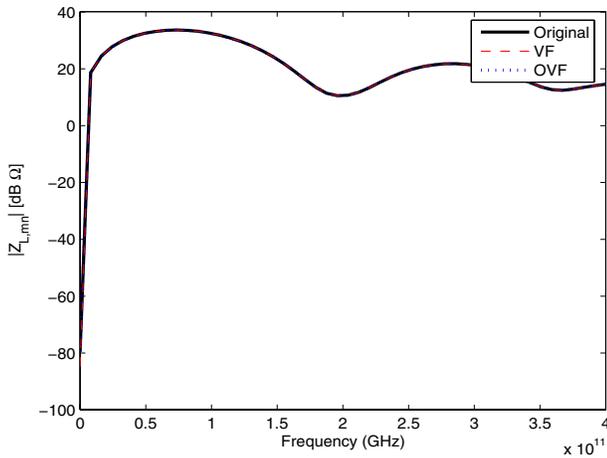


Fig. 2. Mutual impedance $Z_{L,mn}$.

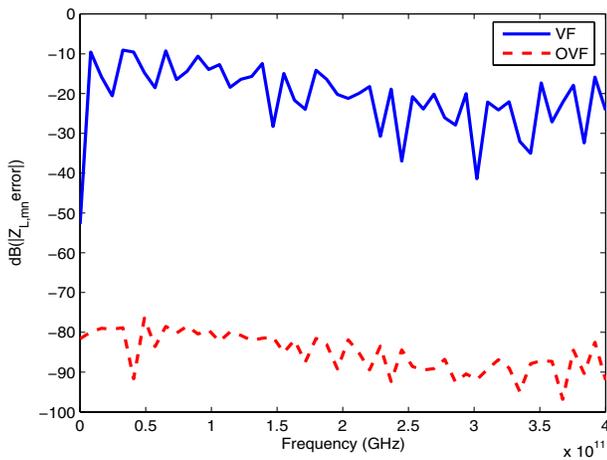


Fig. 3. Fitting accuracy of the mutual impedance $Z_{L,mn}(j\omega)$.

Fig. 2 shows both the rational fitting models which were calculated during the first iteration; visually no real difference can be distinguished. However, Fig. 3 shows the corresponding error; as clearly seen the OVF technique gives remarkably superior results to the classical VF algorithm for this example. It should however be noted that VF and OVF eventually converge to more accurate results if more iterations are performed.

It is also significant comparing the RMS error of the two techniques after one iteration. Table I shows the

TABLE I
GLOBAL FITTING ERROR (TEST IV-A)

VF	OVF
0.1535	7.1792e-005

computed RMS errors: the OVF technique confirms its superiority over the standard VF approach.

This result is extremely important in the case of broadband PEEC modeling because, in this framework, a large number of frequency responses need to be fitted.

B. Transmission lines modeling

In the second test the transmission line referred as Line 2 in [24] has been considered. It is a two-conductor transmission line with frequency dependent per unit length parameters. VF and OVF fitting techniques have been used to obtain a rational approximation of admittance matrix \mathbf{Y} in the frequency range 0 Hz-10 GHz. It can be shown that the advantage of OVF with respect to VF becomes more pronounced if the real part of the starting poles is chosen non-negligible. Therefore, the initial set of 150 poles are now chosen as real and equidistantly spread over the real axis. Fig. 4 shows the accuracy of the approximation obtained by using VF and OVF for the Y_{11} and Y_{12} matrix entry after performing 2 iterations : again, OVF provides considerably better results over the broadband frequency range with respect to VF.

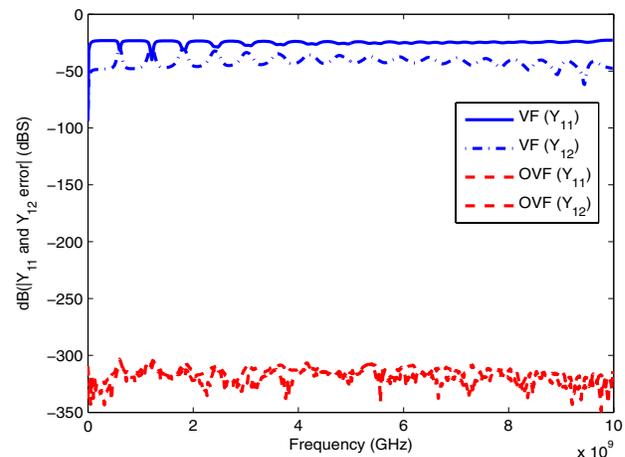


Fig. 4. Fitting accuracy of Y_{11} and Y_{12} .

Table II shows the computed RMS errors for the Y_{12} matrix entry in terms of iteration count. Clearly, OVF was able to calculate a fitting model with an accuracy level up to machine precision in 2 iterations. The classical VF

TABLE II
GLOBAL FITTING ERROR Y_{12} (TEST IV-B)

Iteration	VF	OVF
1	9.0290e-003	5.7132e-012
2	6.6920e-003	2.1620e-016
3	5.0891e-003	2.2681e-017
4	1.1393e-003	1.8622e-016
5	2.0187e-014	1.8317e-016
6	1.5963e-016	2.4962e-016

method needs at least 6 iterations to provide comparable results. This shows that a reduction of 66,67% in computation time can be achieved. Analogous results are obtained for Y_{11} .

V. CONCLUSIONS

This paper has presented a comparative study of the Vector Fitting and Orthonormal Vector Fitting techniques for rational approximation of frequency responses. The numerical tests have proved that OVF is able to provide more accurate results than VF with the same number of iterations. This leads to significant reduction of the overall rational approximation, especially when a large number of frequency responses need to be fitted and makes OVF especially suited to be combined with other numerical techniques for fast broadband modeling.

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