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(54) **BROADBAND SYSTEM MODELS**

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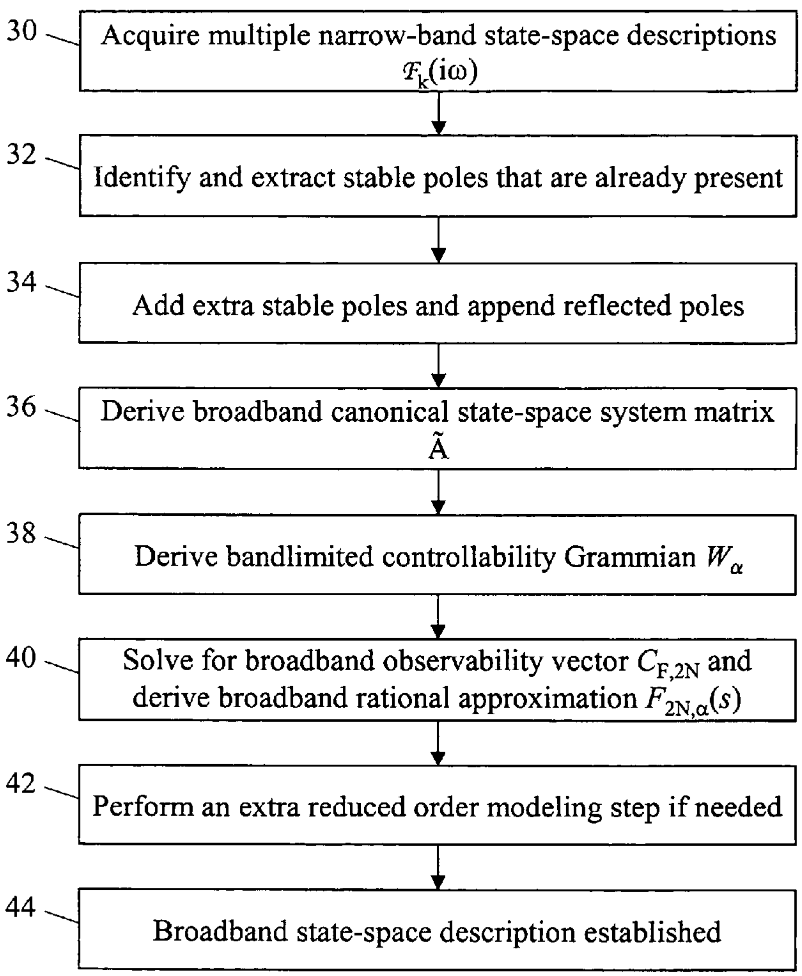
(57) **ABSTRACT**

A method of concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system, each model being descriptive of the system's operational characteristics over a different respective frequency range, to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models, includes assembling stable poles of matrix representations of the narrowband frequency-domain models together with additional poles satisfying a predetermined criterion, based on band-limited truncated Complete Orthonormal Kautz Bases (COKB) requirements, to derive a canonical modal system matrix, deriving a band-limited controllability Grammian as a function of the canonical modal system matrix. deriving a broadband observability vector as a function of the band-limited controllability Grammian and the canonical modal system matrix, and deriving the single broadband model as a function of the broadband observability vector.



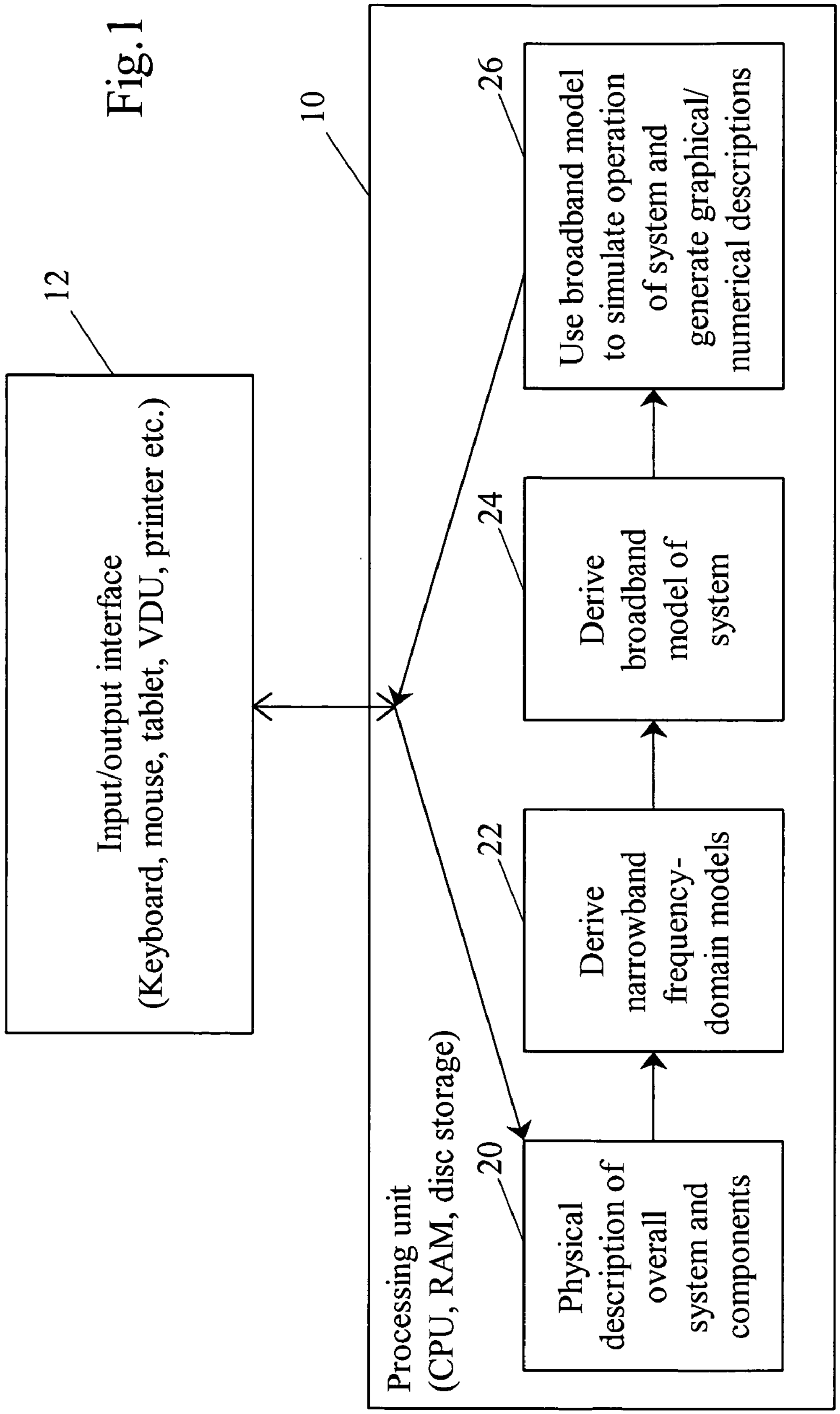
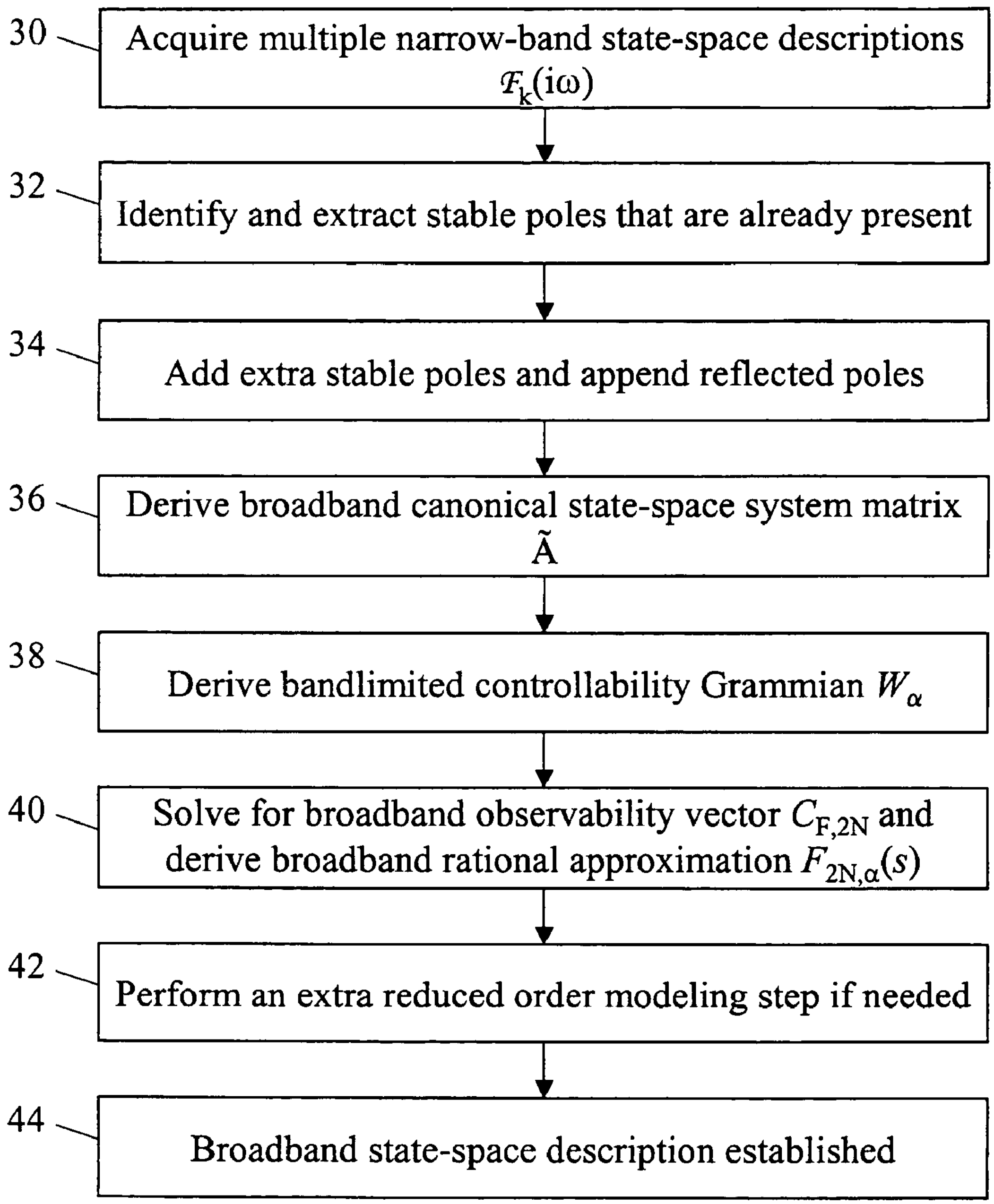

12
Input/output interface
(Keyboard, mouse, tablet, VDU, printer etc.)
10
Processing unit
(CPU, RAM, disc storage)
20
Physical description of overall system and components
22
Derive narrowband frequency-domain models
24
Derive broadband model of system
26
Use broadband model to simulate operation of system and generate graphical/ numerical descriptions

Fig. 1

Fig.2


30
Acquire multiple narrow-band state-space descriptions $\mathcal{F}_k(i\omega)$
32
Identify and extract stable poles that are already present
34
Add extra stable poles and append reflected poles
36
Derive broadband canonical state-space system matrix $\tilde{A}$
38
Derive bandlimited controllability Grammian W_α
40
Solve for broadband observability vector $C_{F,2N}$ and derive broadband rational approximation $F_{2N,\alpha}(s)$
42
Perform an extra reduced order modeling step if needed
44
Broadband state-space description established

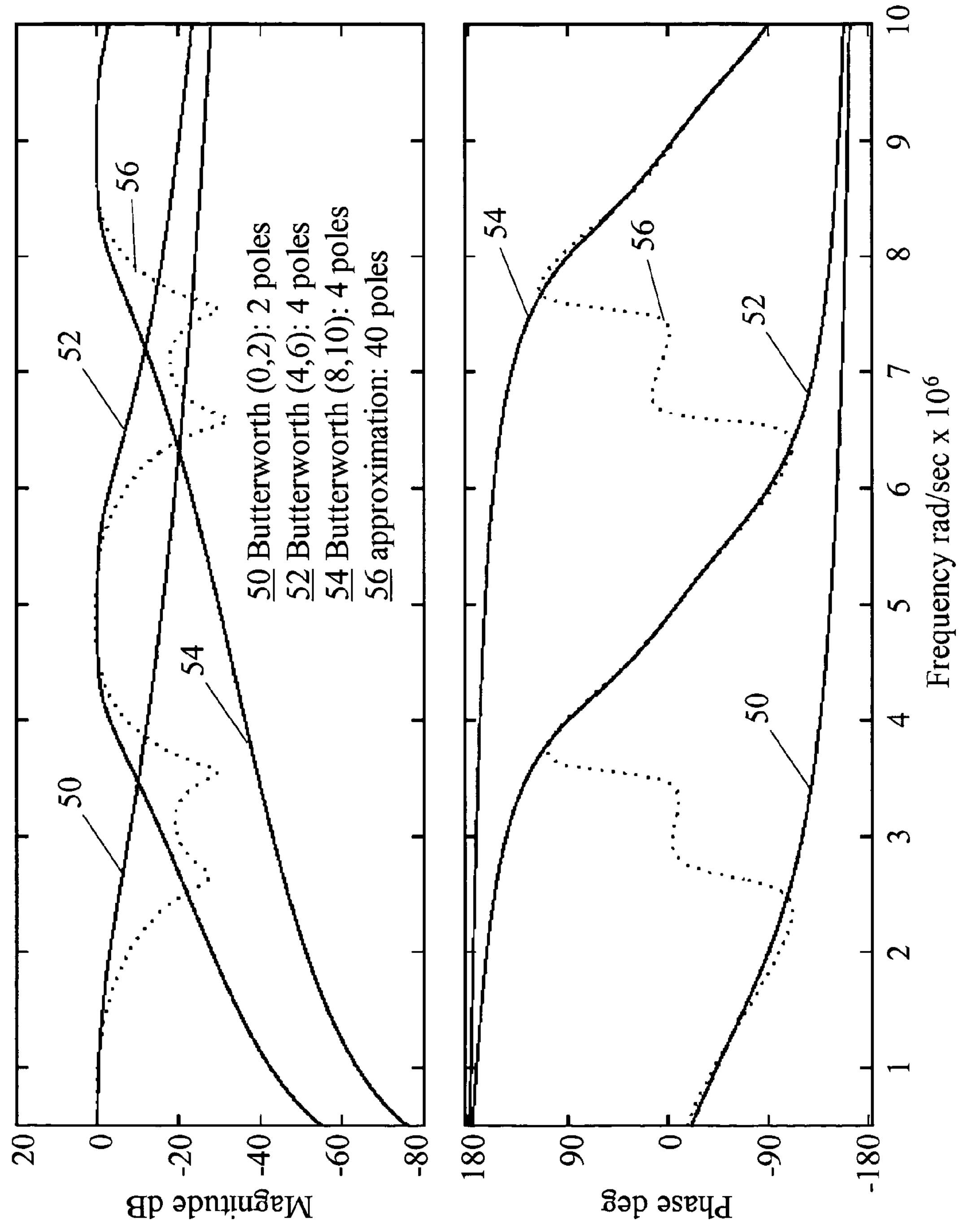
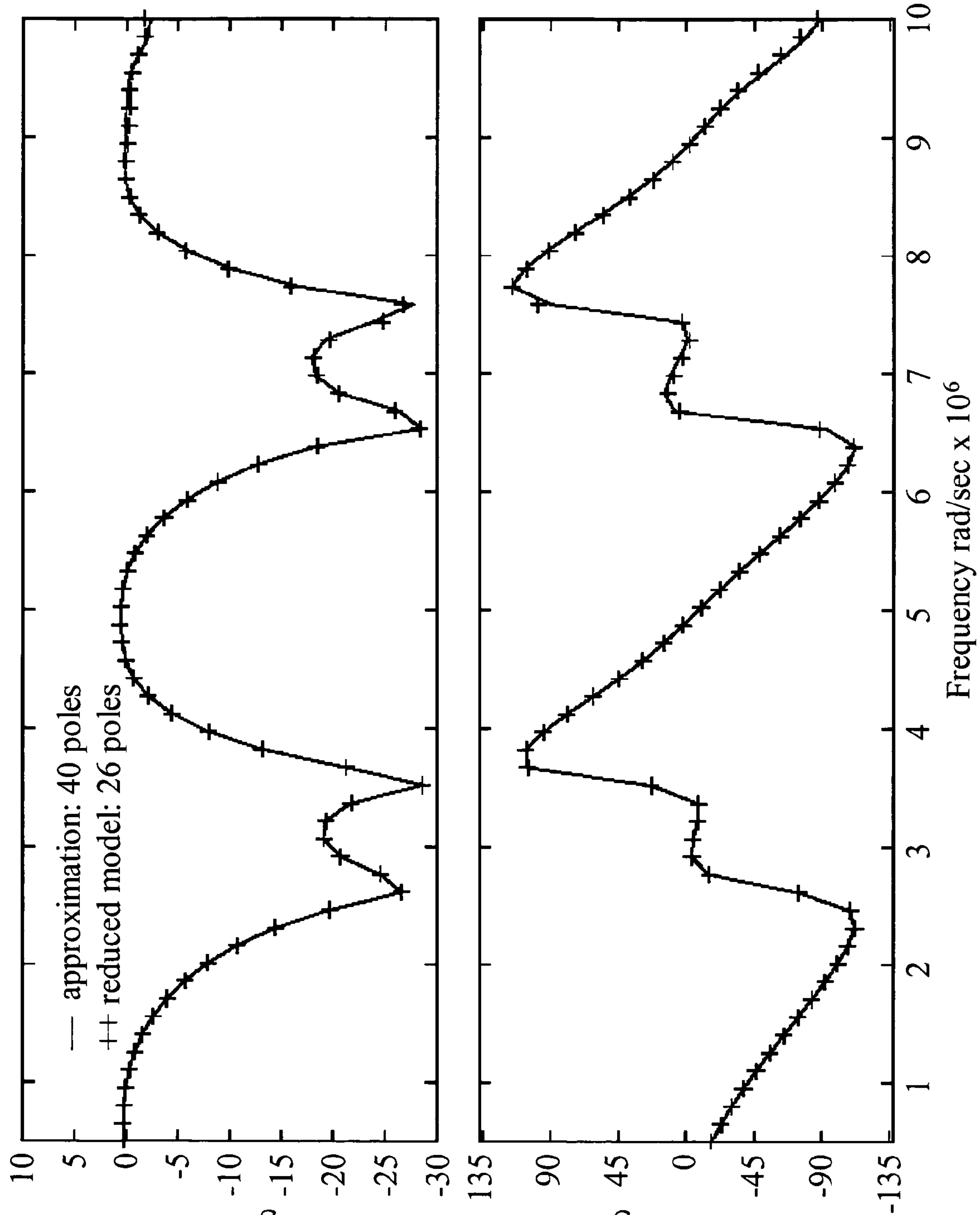

Magnitude dB
20
0
-20
-40
-60
-80
Phase deg
180
90
0
-90
-180
1
2
3
4
5
6
7
8
9
10
Frequency rad/sec x 10^6
50 Butterworth (0,2): 2 poles
52 Butterworth (4,6): 4 poles
54 Butterworth (8,10): 4 poles
56 approximation: 40 poles
50
52
54
56

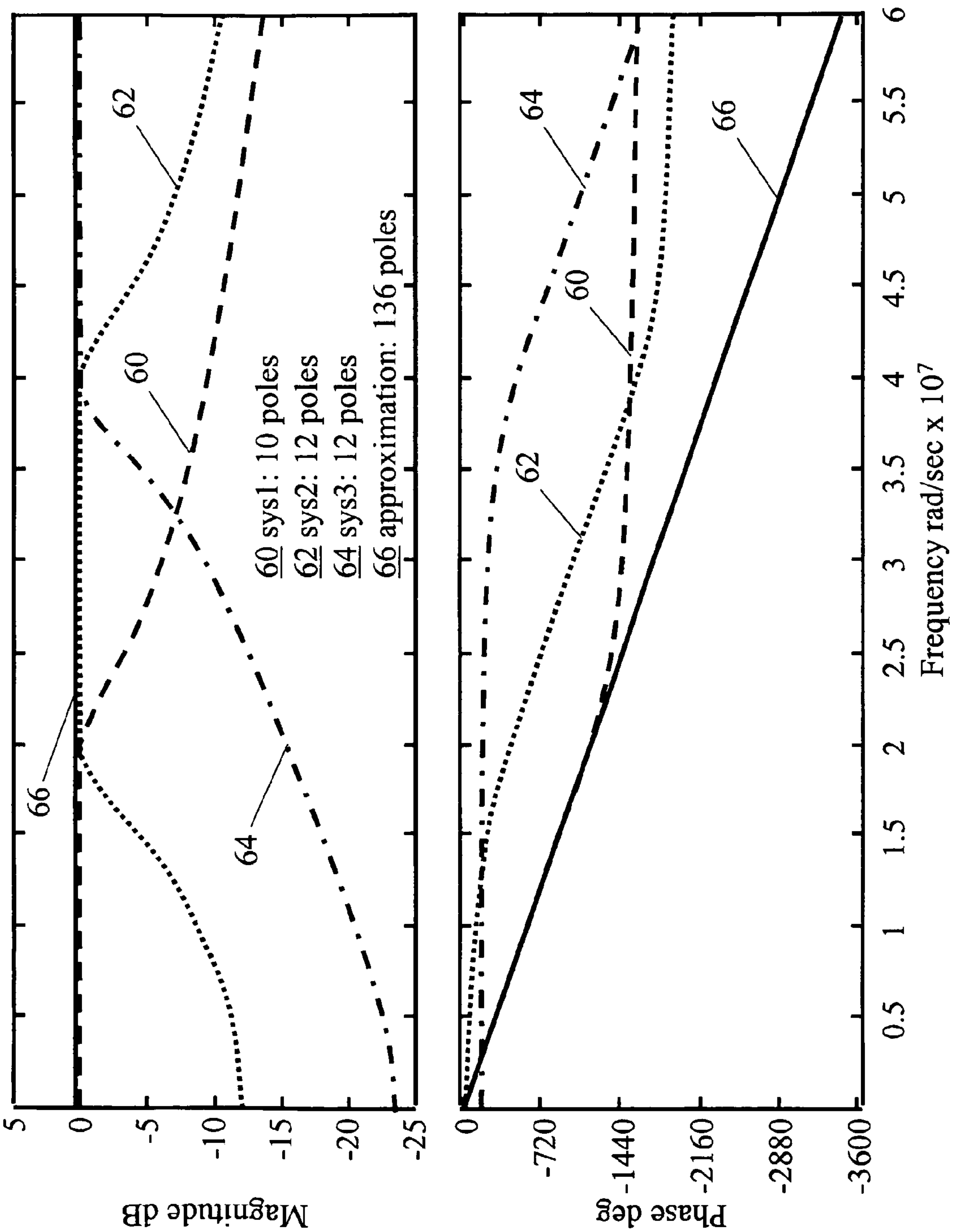
Fig.3



— approximation: 40 poles
++ reduced model: 26 poles
Magnitude dB
10
5
0
-5
-10
-15
-20
-25
-30
Phase deg
135
90
45
0
-45
-90
-135
1
2
3
4
5
6
7
8
9
10
Frequency rad/sec x 10^6

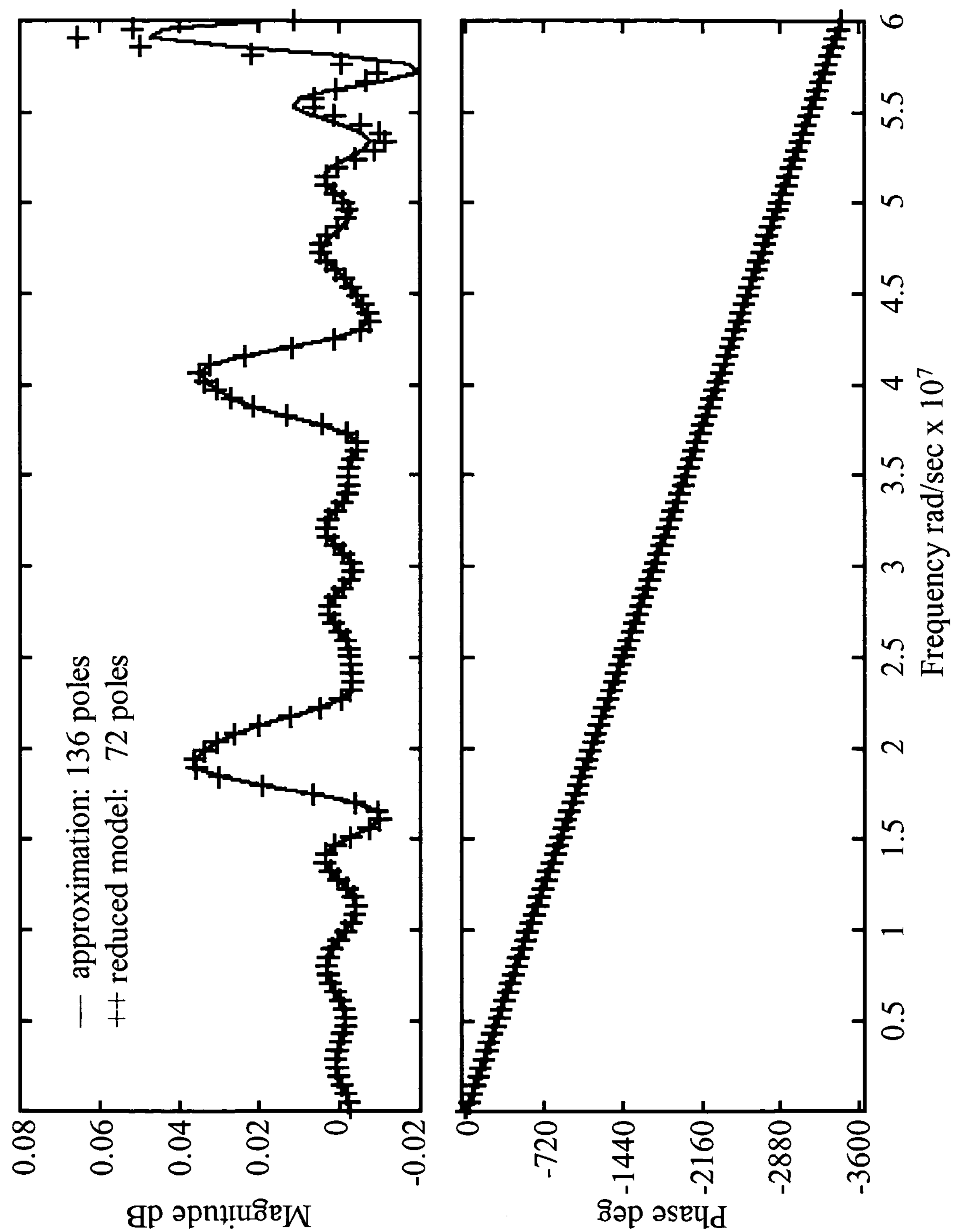
Fig.4

Fig.5



Magnitude dB
5
0
-5
-10
-15
-20
-25
60 sys1: 10 poles
62 sys2: 12 poles
64 sys3: 12 poles
66 approximation: 136 poles
60
62
64
66
Phase deg
0
-720
-1440
-2160
-2880
-3600
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Frequency rad/sec x 10^7

Fig.6


— approximation: 136 poles
++ reduced model: 72 poles
Magnitude dB
0.08
0.06
0.04
0.02
0
-0.02
Phase deg
0
-720
-1440
-2160
-2880
-3600
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Frequency rad/sec x 10^7

BROADBAND SYSTEM MODELS

[0001] This invention relates to broadband system models, and particularly though not exclusively to methods and apparatus of concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models.

BACKGROUND ART

[0002] The use of system simulation techniques to design and analyse complex dynamic systems, incorporating mathematical descriptions of the characteristics of component parts of the systems, has become increasingly widespread. Examples include automotive and aerospace products, and electronic products such as mobile telephones and domestic receivers for satellite TV transmissions. This trend has been accompanied by an increase in the dynamic range over which such simulation techniques are required accurately to model the systems' behaviour. For example, some electronic circuits need to be designed for predictable operation over a frequency range from d.c. to 10 GHz or even 100 GHz. These systems are typically of a kind known as linear time-invariant (LTI), meaning that they comply with the principle of superposition and that time shifts in the input signal produce equal time shifts in the output signal.

[0003] Known simulation techniques include adaptive frequency sampling (AFS) ("Adaptive frequency sampling algorithm for fast and accurate S-parameter modelling of general planar structures", T. Dhaene, J. Ureel, N. Fache & D. De Zutter, *Proceedings of the IEEE International Microwave Symposium,* May 1995) and narrow-band information methods ("Accurate computation of wide-band response of electromagnetic systems using narrowband information", K. Kottapali, T. K. Sarkar, Y. Hua, E. K. Miller & G. J. Burke, *IEEE Trans. Microwave Theory Techn,* April 1991; and "Efficient frequency-domain modelling and circuit simulation of transmission lines", L. M. Silveira, I. M. Elfadel, J. K. White, M. Chilukuri & K. S. Kundert, *IEEE Trans. Components, Packaging and Manufacturing Technol., Part B: Advanced Packaging,* vol. 17, no. 4, pp. 505-513, November 1994). Use of these techniques in actual circuit simulations generates multiple piecewise rational models, valid only over relatively small frequency ranges. However, further processing for example by means of SPICE (Simulation Program with Integrated Circuit Emphasis) netlists requires a single "global" rational model that is valid over the overall frequency range.

[0004] Prior proposals for solving the problem of building global broadband models based on multiple narrow-band rational approximations include:

[0005] 1. A straightforward "brute-force" system identification approach, as described in "Identifying S-parameter models in the Laplace domain for high frequency multiport linear networks", A. Verschueren, Y. Rolain, R. Vuerinckx & G. Vandersteen, *Microwave Symposium Digest,* 1998 *IEEE MTT-S International,* vol. 1, June 1998. This really consists of a mere re-sampling over the overall frequency range, without taking advantage of the known characteristics (poles, zeros, gains) of the piecewise rational models, and thereby loses track of pertinent information. This technique has the following practical disadvantages: it is a brute-force, computationally expensive method; there is no use of a-priori knowledge (poles/zeros, poles/residues); there are numerical stability issues if the system being modelled has a large number of poles; over-sampling (too many data samples) can occur; over-modelling (too many poles) is likely.

[0006] 2. Complex Frequency Hopping (CFH), described in "Analysis of interconnect networks using complex frequency hopping (CFH)", E. Chiprout & M. S. Nakhla, *IEEE Trans. Computer-Aided Design,* vol. 14, no. 2, pp. 186-200, February 1995. This is a heuristic technique that combines a relatively small set of dominant poles of multiple narrowband frequency ranges into one global system model. CFH uses the concept of "moment matching" to obtain a lower-order multi-point Pade approximation. This technique likewise has practical disadvantages: it is based on heuristics and hard to apply automatically; only a subset of poles is considered; it has limited accuracy and it is not possible to estimate the accuracy of the approximating model generated.

DISCLOSURE OF INVENTION

[0007] According to one aspect of this invention there is provided a method of concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system, each model being descriptive of the system's operational characteristics over a different respective frequency range, to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models, comprising the steps of:

[0008] assembling stable poles of matrix representations of the narrowband frequency-domain models together with additional poles satisfying a predetermined criterion, based on band-limited truncated Complete Orthonormal Kautz Bases (COKB) requirements, to derive a canonical modal system matrix;

[0009] deriving a band-limited controllability Grammian as a function of said canonical modal system matrix;

[0010] deriving a broadband observability vector as a function of said band-limited controllability Grammian and said canonical modal system matrix; and

[0011] deriving said single broadband model as a function of said broadband observability vector.

[0012] According to another aspect of this invention there is provided apparatus for concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system, each model being descriptive of the system's operational characteristics over a different respective frequency range, to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models, comprising:

[0013] a matrix generator for assembling stable poles of matrix representations of the narrowband frequency-domain models together with additional poles satisfying a predetermined criterion, based on band-limited truncated Complete Orthonormal

Kautz Bases (COKB) requirements, to derive a canonical modal system matrix;

[0014] a Grammian generator for deriving a band-limited controllability Grammian as a function of said canonical modal system matrix;

[0015] a vector generator for deriving a broadband observability vector as a function of said band-limited controllability Grammian and said canonical modal system matrix; and

[0016] a model generator for deriving said single broadband model as a function of said broadband observability vector.

[0017] The invention accomplishes its purpose in part by making use of a novel, band-limited variant of Complete Orthonormal Kautz Bases (COKB). These Bases have been previously described for use in continuous-time system modelling ("Orthonormal basis functions for modelling continuous-time systems", H. Akcay & B. Ninness, *Signal Processing,* vol. 77, no. 3, pp. 261-274, September 1999) and system identification ("System identification using Kautz models", B. Wahlberg, *IEEE Trans. Aut. Control,* vol. 39, pp. 1276-1281, 1994). However, a major drawback of the COKB as previously described is that full frequency-domain knowledge is needed in order to calculate the pertinent Hardy space scalar products, whereas most often in practice essentially band-limited functions and systems are encountered. The inventors hereof have succeeded in developing a novel, truncated implementation of COKB, allowing the derivation of orthonormal band-limited Kautz sequences.

[0018] According to a further aspect of this invention, therefore, there is provided a method of modelling a linear time-invariant system, wherein a model of the system is constructed incorporating a set of stable poles generated using an α-band-limited truncated Complete Orthonormal Kautz Bases (COKB) sequence defined by

$$\Xi_n(s) = \sqrt{-2\mathcal{R}(p_n)}\,\frac{\alpha(s+\alpha)}{\alpha^2 s - p_n(s^2+\alpha^2)}\prod_{k=0}^{n-1}\left(\frac{\alpha^2 s + \overline{p_k}(s^2+\alpha^2)}{\alpha^2 s - p_k(s^2+\alpha^2)}\right)$$

$$n = 0.1.2\ \ldots$$

[0019] where R indicates the real part of a complex expression, Π indicates the product of the specified series of factors, α is the overall bandwidth, s is the complex frequency, and p_n are the original poles.

BRIEF DESCRIPTION OF DRAWINGS

[0020] A method and apparatus in accordance with this invention, for simulating operation of an LTI system such as an electronic circuit, will now be described, by way of example, with reference to the accompanying drawings, in which:

[0021] **FIG. 1** is a block diagram of apparatus for simulating operation of an LTI system using the present invention;

[0022] **FIG. 2** is a flow chart of a procedure implemented in the apparatus of **FIG. 1**;

[0023] **FIG. 3** is a Bode diagram of the frequency response of a set of Butterworth filters modelled using the invention, showing simulation results provided by an initial approximation;

[0024] **FIG. 4** is a Bode diagram of the frequency response of the set of Butterworth filters of **FIG. 3**, showing simulation results provided after a reduced order modelling step;

[0025] **FIG. 5** is a Bode diagram of the frequency response of a pure delay transfer function modelled using the invention, showing simulation results provided by an initial approximation; and

[0026] **FIG. 6** is a Bode diagram of the frequency response of the pure delay transfer function of **FIG. 5**, showing simulation results provided after a reduced order modelling step.

DETAILED DESCRIPTION

[0027] The invention enables broadband system models to be assembled from two or more narrowband frequency-domain models of a linear time-invariant (LTI) system. A linear system is one to which the principle of superposition applies, i.e. the output of the system in response to two different stimuli applied simultaneously is equal to the sum of the system outputs in response to the two stimuli applied individually. Thus if:

$$x_1 \rightarrow y_1 \text{ and } x_2 \rightarrow y_2$$

[0028] where x_1 and x_2 are system inputs, y_1 and y_2 are the system outputs, and → indicates "results in the response", then in a linear system:

$$ax_1+bx_2 \rightarrow ay_1+by_2$$

[0029] where a and b are arbitrary constants.

[0030] A system is time-invariant if time shifts in the input signal produce equal time shifts in the output signal. Thus if:

$$x(t) \rightarrow y(t)$$

[0031] then in a time-invariant system, for any time shift t_0:

$$x(t-t_0) \rightarrow y(t-t_0)$$

[0032] Examples of LTI systems are found in a variety of disciplines: electronic circuits such as satellite microwave receivers, radio-frequency and microwave circuits; mechanical systems such as oscillators (e.g. vehicle suspensions and other sprung systems) and disk drives; electrical power systems, such as transformers; computer systems; biological systems; and economic systems.

[0033] For convenience an example implementation of the invention will be described in the context of electronic circuit design, using apparatus as shown in **FIG. 1** for simulating operation of an electronic circuit. However, the invention is equally applicable to simulating the operation of any other kind of LTI system, including those mentioned above.

[0034] Referring to **FIG. 1**, the apparatus comprises a processing unit **10** and a user input/output interface unit **12**. The processing unit **10** includes a central processing unit (CPU), random-access memory (RAM), hard disc storage and associated circuitry to enable the CPU to implement procedures in accordance with software program instructions stored in the RAM, and to interact with the interface unit **12** to receive input from the user and display the results of the procedures. The interface unit **12** typically comprises a visual-display unit (VDU), keyboard, mouse and/or tablet or similar pointing device, and a printer or other hard-copy output device.

[0035] In preparing to perform a system simulation, the apparatus receives, via the interface unit **12**, a physical description of the system at step **20**, for example a list of components of an electronic circuit, their operating characteristics (e.g. resistance, capacitance, gain as a function of frequency, etc.), their interconnection and other details of the circuit layout. At step **22** the apparatus derives a plurality of narrowband frequency-domain models of the system's operation. The number of models will depend in particular on the frequency range over which the operation of the system is to be simulated. These models may conveniently be in state-space format, comprising (in generalised terms): a state equation of the form

$$x'=Ax+Bu$$

[0036] where x' (bold type indicates a matrix or vector) is the derivative of the system's state vector with respect to time, A is the system matrix, B is the input matrix and u is the input; and an output equation of the form

$$y=Cx+Du$$

[0037] where y is the output, C is the output matrix, and D is the feedforward term.

[0038] At step **24** the plurality of narrowband models is used to derive a single broadband model of the system's behaviour over the entire frequency range of interest, as described in more detail herein. At step **26** the broadband model is used to simulate operation of the system and generate output data that describes such operation. These output data may comprise, for example, graphical displays of circuit operating characteristics, such as Bode diagrams, Smith charts and pole-zero diagrams, and numerical descriptions such as parameter values for formulae that summarise the system's properties. The output data are supplied to the user via the interface unit **12**, and may be used to understand the operating characteristics of the simulated system, compare its behaviour with that which is desired, refine the design of the system, and provide data to control manufacturing processes to assemble a practical implementation of the system.

[0039] The operation of the apparatus in relation to steps **20** and **22** is conventional, and need not be described further here. The derivation of a single broadband model proceeds as summarised in **FIG. 2**. At step **30** a plurality M of piecewise, narrowband state-space descriptions $F_k(i\omega)$ is acquired, using conventional techniques as described above. These descriptions may be summarised as

$$F(i\omega)=F_k(i\omega)=C_k^T(i\omega I-A_k)^{-1}B_k \quad 0\leqq_k\leqq|\omega|\leqq\beta_k \quad k=1 \ldots M \tag{1}$$

[0040] with $0<\beta_k\leqq\gamma_{k+1}$ and $\beta_M=\alpha$, where α is the overall bandwidth of the desired single broadband model and ω is the radial frequency. The objective is to obtain a single rational approximation that well matches all the piecewise, band-limited functions $F_k(i\omega)$ over the frequency range $[-\alpha, \alpha]$. As explained below, this rational approximation takes the form

$$F_{2N,\alpha}(s)=\sum_{n=0}^{2N-1}\langle F\mid\tilde{\xi}_n\rangle_\alpha\tilde{\xi}_n(s)=\tilde{C}_{F,2N}^T\left(sI-\tilde{A}\right)^{-1}\tilde{B} \tag{2}$$

[0041] To obtain this overall rational approximation $F_{2N,\alpha}(s)$, an appropriate stable pole segment that is already present in the piecewise data is first selected, at step **32**. A description of how to perform this individual step is given in the above-mentioned paper by Dhaene, Ureel, Fache & De Zutter. A first and straightforward requirement for accomplishing it is to include the set of all the stable poles of the matrices A_k in the $\{q_k\}$ pole sequence of the broadband representation. To further enhance the dynamic range, this pole segment is extended at step **34** by a truncated sequence of other stable poles satisfying the Müntz-Szász condition (see "Equivalent formulations of the Muntz-Szasz completeness condition for systems of complex exponentials", L. Knockaert, *Journal of The Franklin Institute,* vol. 339, no. 1, pp 103-109, January 2002). One possibility is a sequence of equal Laguerre poles $\{-\alpha;\ k=1, \ldots, L\}$; another possibility, particularly where it is desired to avoid degeneracy problems related to coinciding poles, is a sequence of the form $\{-k\alpha/(k+1);\ k=1, \ldots, L\}$; further details of such sequences are given in "On orthonormal Muntz-Laguerre filters", L. Knockaert, *IEEE Trans. on Signal Processing,* vol. 49, no. 4, pp. 790-793, April 2001. The full pole segment being assembled at step **34** is obtained by appending the reflected pole set $\{\alpha^2/q_k\}$. The reflected poles enable compliance with the α-band-limited truncated Complete Orthonormal Kautz Bases (band-limited truncated COKB) requirements that have been derived for the first time by the inventors hereof:

$$\Xi_n(s)=\sqrt{-2\mathcal{R}(p_n)}\,\frac{\alpha(s+\alpha)}{\alpha^2 s-p_n(s^2+\alpha^2)}\prod_{k=0}^{n-1}\left(\frac{\alpha^2 s+\overline{p_k}(s^2+\alpha^2)}{\alpha^2 s-p_k(s^2+\alpha^2)}\right) \tag{3}$$

$$n=0.1.2\ \ldots$$

[0042] where R indicates the real part of a complex expression, Π indicates the product of the specified series of factors, α is the overall bandwidth, s is the complex frequency, and p_n are the original poles.

[0043] At step **36** a broadband canonical modal state-space system matrix $\tilde{A}$ is constructed, by combining all the stable poles $\{q_k\}$ of the matrices A_k (system matrices) of the state-space representations $F_k(i\omega)$, plus the stable poles $\{-k\alpha/(k+1);\ k=1, \ldots, L\}$, plus the reflected poles $\{\alpha^2/q_k\}$ appended in step **34**, which depend only on the previously-generated full pole sequences (of step **32**).

[0044] At step **38** a band-limited controllability Grammian W_α is derived from the piecewise, narrowband state-space descriptions $F(i\omega)$ acquired at step **30**, by way of narrowband scalar products as described below. The controllability Grammian is a known concept in state-space description of systems, providing information on whether there exists a system input for any initial state of the system that will bring it to some other defined state in a defined time interval. The band-limited controllability Grammian W_α is derived at step **38** according to the expression

$$W_\alpha=\langle(i\omega I-\tilde{A})^{-1}\tilde{B}|[(i\omega I-\tilde{A})^{-1}\tilde{B}]^T\rangle_\alpha \tag{4}$$

[0045] where $\tilde{A}$ is the 2N×2N broadband state-space system matrix obtained at step **36** and $\tilde{B}$ is a column vector of length 2N consisting of only ones.

[0046] A narrowband (α-band-limited) scalar product of LTI state-space transfer functions such as that in expression (4) above, represented by the notation "$\langle.|.\rangle_\alpha$", can be efficiently computed by using the relationship

$$\langle F_1 \mid F_2\rangle_\alpha = -\frac{1}{\pi}C_{12}^T\text{arccot}\left(\frac{A_{12}}{\alpha}\right)B_{12} \tag{5}$$

[0047] where F_1 and F_2 are the state-space transfer functions whose scalar product is required (for example of the form $[C_1^T(i\omega I-A_1)^{-1}B_1]$ and $[C_2^T(i\omega I-A_2)^{-1}B_2]$), C^T indicates the transpose of the matrix C, arccot is the arccotangent function, and A_{12}, B_{12} and C_{12} are matrices derived from these functions:

$$C_{12}=\begin{pmatrix}0\\C_2\end{pmatrix}\quad B_{12}=\begin{pmatrix}B_1\\0\end{pmatrix}\quad A_{12}=\begin{pmatrix}A_1 & 0\\ -B_2C_1^T & -A_2\end{pmatrix} \tag{6}$$

[0048] In applying expression (5) to the evaluation specifically of expression (4), C_1^T and C_2^T each take on the value of the diagonal identity matrix. The computation of the expression arccot(A), where A is any real matrix, can be accomplished using a method devised by B. N. Parlett and described in "Matrix Computations" by G. H. Golub & C. F. Van Loan, The John Hopkins University Press, 1996, section 11.1, pp. 380-387.

[0049] The band-limited controllability Grammian W_α derived at step **38** is then used in step **40** to evaluate a broadband observability vector $C_{F,2N}$, according to the following relationship

$$\tilde{C}_{F,2N} = W_\alpha^{-1}\mathcal{R}\left\{\left\langle(i\omega-\tilde{A})^{-1}\tilde{B}\,\middle|\,F(i\omega)\right\rangle_\alpha\right\} \tag{7}$$

[0050] where R indicates the real part of the complex expression between braces {}. The scalar product in this complex expression can be evaluated using the relationship (5) as described above.

[0051] The required rational approximation of a single, broadband state-space model of the system to be simulated can then be derived using expression (2) given above.

[0052] As a consequence of the manner of construction of the system matrix $\tilde{A}$ at step **36**, the function for $F_{2N,\alpha}(s)$ on the right-hand side of expression (2) may have an excessively large model order (number of poles). Therefore at optional step **42** the state-space representation of $F_{2N,\alpha}(s)$ may be input into a reduced order modelling algorithm such as the Laguerre-SVD algorithm, described in "Laguerre-SVD reduced order modeling", L. Knockaert & D. De Zutter, *IEEE Trans. Microwave Theory Techn.,* vol. 48, no. 9, pp. 1469-1475, September 2000, to obtain at step **44** a final broadband state-space model of sufficiently low order.

[0053] An example of the invention applied to the modelling of a system comprising three Butterworth filters will be described: the first filter is two-pole, low-pass with cutoff $\omega_c=2\times10^6$ rad/s; the second is four-pole, band-pass over the frequency range 4×10^6 rad/s≦ω≦6×10^6 rad/s; and the third is four-pole, band-pass over the higher range 8×10^6 rad/s≦ω≦10^7 rad/s. The purpose is to find a single (global) state-space model which is sufficiently close to each of the three filters considered individually in its respective frequency band. The global bandwidth $\alpha=10^7$ rad/s and the number of stable poles is 10. Adding 10 approximate Laguerre poles {−α/2, −2α/3, . . . , −10 α/11} and reflecting the poles by means of the transformation α^2/p (step **34**) results in a total of 40 system poles to include in the global model. Applying the procedure described with reference to steps **38** and **40**, using the arccot relationship (4), results in a value for the observability vector $C_{F,2N}$, and an initial global state-space description. A Bode diagram of this description is shown in **FIG. 3**, where the continuous lines **50**, **52** and **54** indicate the characteristics of the component Butterworth filters, and the dotted line **56** indicates the characteristics of the global state-space model. After a Laguerre-SVD reduced order modelling step **42** (**FIG. 2**), an accurate reduced-order model with 26 system poles is obtained, with characteristics as shown in the Bode diagram of **FIG. 4** (which covers a more restricted range of magnitude and phase than **FIG. 3**).

[0054] Sometimes a Neville-type rational interpolation procedure such as the Bulirsch-Stoer algorithm (described in "Numerical Recipes in Fortran, The Art of Scientific Computing", W. H. Press, S. A. Teukolsky, W. T. Vetterling & B. P. Flannery, 2nd Ed., Cambridge University Press, 1992) can be used to find a rational function that is close to a tabulated function over a certain frequency range. A convenient version of this algorithm, similar though not identical to the one presented in "An efficient adaptive frequency sampling algorithm for model-based parameter estimation as applied to aggressive space mapping", R. Lehmensiek & P. Meyer, *Microwave Opt. Techn. Lett.,* vol. 24, no. 1, pp. 71-78, January 2000, is as follows:

> **[0055]** Consider a frequency response table h={H_1, $\bar{H}_1$, K, H_N, $\bar{H}_N$} at the complex frequencies {$s_1=i\omega_1$, $s_2=-i\omega_1$, . . . , $s_{2N-1}=i\omega_N$, $s_{2N}=-i\omega_N$} with $0<\omega_1<\ldots<\omega_N<\infty$. Then a real-ratio function $R_{2N}(s)=a_{2N}(s)/b_{2N}(s)$ with N poles and N-1 zeros such that $R_{2N}(s_k)=h_k$ can be constructed by the Neville-type algorithm

$$a_k(s)=\sigma_k a_{k-1}(s)+(s-s_{k-1})a_{k-2}(s) \tag{8}$$

$$b_k(s)=\sigma_k b_{k-1}(s)+(s-s_{k-1})b_{k-2}(s) \tag{9}$$

[0056] with initial values $a_0=0$, $a_1=h_1$, $b_1=b_0=1$. The value for σ_k is found by requiring that $h_k=a_k(s_k)/b_k(s_k)$, i.e.

$$\sigma_k = (s_{k-1}-s_k)\frac{h_k b_{k-2}(s_k)-a_{k-2}(s_k)}{h_k b_{k-1}(s_k)-a_{k-1}(s_k)} \tag{10}$$

[0057] It would be convenient if the above interpolation algorithm also exhibited some extrapolation power, but unfortunately in practice this is rarely the case. To obtain a rational approximation of a given analytic function over a large bandwidth, we therefore need to interpolate over different relatively narrow bands, and afterwards combine the approaches in an overall rational model.

[0058] As an example, consider the pure delay transfer function e^{-st} with $\tau=1$ μs. Applying the Neville-type algorithm on equispaced samples in the bands $0<\omega\leqq 2\times10^7$ rad/s, 2×10^7 rad/s$\leqq\omega\leqq 4\times10^7$ rad/s, and 4×10^7 rad/s$\leqq\omega\leqq 6\times10^7$ rad/s, rational interpolants are obtained with respectively 10, 12 and 12 poles. The global bandwidth $\alpha=6\times10^7$ rad/s and the number of stable poles is 34. Adding 34 approximate Laguerre poles $\{-\alpha/2, -2\alpha/3, \ldots, -34\alpha/35\}$ and reflecting the poles by means of the transformation $\alpha^{2/}p$ (step **34** in **FIG. 2**) results in a total of 136 system poles to process. Applying the procedure described with reference to steps **38** and **40**, using the arccot relationship (4), results in a value for the observability vector $C_{F,2N}$, and an initial global state-space description. A Bode diagram is shown in **FIG. 5**, where the discontinuous lines **60**, **62** and **64** indicate the magnitude and phase characteristics of the three sets of rational interpolants, and the continuous line **66** indicates the characteristics of the global state-space model. (In the magnitude diagram the line **66** has been displaced fractionally upwards from its true position to make the interpolant lines that it overlaps more clearly visible.) After a Laguerre-SVD reduced order modelling step **42** (**FIG. 2**), an accurate reduced-order model with 72 system poles is obtained, with characteristics as shown in the Bode diagram of **FIG. 6**. In comparing **FIGS. 5 and 6** it should be noted that the magnitude diagram in **FIG. 5** embraces a range of +5 to −25 dB, whereas the magnitude range in **FIG. 6** is much more restricted (+0.08 to −0.02 dB) to make subtle variations in magnitude response more evident.

1. A method of concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system, each model being descriptive of the system's operational characteristics over a different respective frequency range, to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models, comprising the steps of:

assembling stable poles of matrix representations of the narrowband frequency-domain models together with additional poles satisfying a predetermined criterion, based on band-limited truncated Complete Orthonormal Kautz Bases (COKB) requirements, to derive a canonical modal system matrix;

deriving a band-limited controllability Grammian as a function of said canonical modal system matrix;

deriving a broadband observability vector as a function of said band-limited controllability Grammian and said canonical modal system matrix; and

deriving said single broadband model as a function of said broadband observability vector.

2. The method of claim 1, including the step of applying a reduced order algorithm to said broadband model to reduce the number of poles.

3. The method of claim 2, wherein said reduced order algorithm is a Laguerre-SVD algorithm.

4. The method of claim 1, wherein the narrowband scalar products of LTI state-space transfer functions are derived in accordance with the expression

$$\langle F_1 \mid F_2\rangle_\alpha = -\frac{1}{\pi}C_{12}^T \operatorname{arccot}\left(\frac{A_{12}}{\alpha}\right)B_{12}$$

where F_1 represents a first state-space transfer function, of the form $[C_1^T(i\omega I-A_1)^{-1}B_1]$;

F_2 represents a second state-space transfer function, of the form $[C_2^T(i\omega I-A_2)^{-1}B_2]$;

α is the total frequency range of the single broadband model;

C^T represents the transpose of a matrix C;

arccot is the arc-cotangent function; and

A_{12}, B_{12} and C_{12} are matrices derived from the functions:

$$C_{12} = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} \quad B_{12} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad A_{12} = \begin{pmatrix} A_1 & 0 \\ -B_2C_1^T & -A_2 \end{pmatrix}$$

5. The method of claim 1, wherein the band-limited controllability Grammian W_α is derived in accordance with the expression

$$W_\alpha = \langle (i\omega I-\tilde{A})^{-1}\tilde{B} \| [(i\omega I-\tilde{A})^{-1}\tilde{B}]^T \rangle_\alpha$$

where $\tilde{A}$ is a 2N×2N broadband state-space system matrix and $\tilde{B}$ is a column vector of length 2N consisting of only ones.

6. The method of claim 1, wherein the broadband observability vector is derived in accordance with the expression

$$\tilde{C}_{F,2N} = W_\alpha^{-1}\mathrm{R}\left\{\left\langle (i\omega-\tilde{A})^{-1}\tilde{B} \mid F(i\omega)\right\rangle_\alpha\right\}$$

where W_α is the band-limited controllability Grammian;

R indicates the real part of the complex expression between braces {};

$\tilde{A}$ is a 2N×2N broadband state-space system matrix;

$\tilde{B}$ is a column vector of length 2N consisting of only ones; and

$<.|.>_\alpha$ indicates an ax-band-limited scalar product of state-space transfer functions.

7. The method of claim 1, wherein the broadband model is derived in accordance with the expression

$$F_{2N,\alpha}(s) = \sum_{n=0}^{2N-1} \left\langle F \mid \tilde{\xi}_n\right\rangle_\alpha \tilde{\xi}_n(s) = \tilde{C}_{F,2N}^T\left(sI-\tilde{A}\right)^{-1}\tilde{B}$$

where $C^T_{F,2N}$ is the broadband observability vector;

$\tilde{A}$ is a 2N×2N broadband state-space system matrix; and

$\tilde{B}$ is a column vector of length 2N consisting of only ones.

8. The method of claim 1, wherein a set of stable poles is generated using an α-band-limited truncated Complete Orthonormal Kautz Bases (COKB) sequence defined by

$$\Xi_n(s) = \sqrt{-2\Re(p_n)}\,\frac{\alpha(s+\alpha)}{\alpha^2 s - p_n(s^2+\alpha^2)}\prod_{k=0}^{n-1}\left(\frac{\alpha^2 s + \overline{p_k}(s^2+\alpha^2)}{\alpha^2 s - p_k(s^2+\alpha^2)}\right)$$

$$n = 0, 1, 2\ldots$$

where R indicates the real part of a complex expression, Π indicates the product of the specified series of factors, α is the overall bandwidth, s is the complex frequency, and p_n are the original poles.

9. Apparatus for concatenating a plurality of narrowband frequency-domain models of a linear time-invariant (LTI) system, each model being descriptive of the system's operational characteristics over a different respective frequency range, to derive a single broadband model that describes the system's operational characteristics over the total frequency range encompassed by the narrowband models, comprising:

a matrix generator for assembling stable poles of matrix representations of the narrowband frequency-domain models together with additional poles satisfying a predetermined criterion, based on band-limited truncated Complete Orthonormal Kautz Bases (COKB) requirements, to derive a canonical modal system matrix;

a Grammian generator for deriving a band-limited controllability Grammian as a function of said canonical modal system matrix;

a vector generator for deriving a broadband observability vector as a function of said band-limited controllability Grammian and said canonical modal system matrix; and

a model generator for deriving said single broadband model as a function of said broadband observability vector.

10. A method of modelling a linear time-invariant (LTI) system, wherein a model of the system is constructed incorporating a set of stable poles generated using an a-band-limited truncated Complete Orthonormal Kautz Bases (COKB) sequence defined by

$$\Xi_n(s) = \sqrt{-2\Re(p_n)}\,\frac{\alpha(s+\alpha)}{\alpha^2 s - p_n(s^2+\alpha^2)}\prod_{k=0}^{n-1}\left(\frac{\alpha^2 s + \overline{p_k}(s^2+\alpha^2)}{\alpha^2 s - p_k(s^2+\alpha^2)}\right)$$

$$n = 0, 1, 2\ldots$$

where R indicates the real part of a complex expression, Π indicates the product of the specified series of factors, α is the overall bandwidth, s is the complex frequency, and p_n are the original poles.

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