ROBUST IDENTIFICATION OF TRANSIENT PORT RESPONSES USING TIME DOMAIN ORTHONORMAL VECTOR FITTING

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Abstract: A robust technique is presented for parametric identification of linear time-invariant systems in the time domain. Based on a Sanathanan-Koerner iteration, the transfer function coefficients are calculated iteratively by minimizing a weighted linear cost function. The time-domain equivalent of the Muntz-Laguerre basis is used to improve the numerical conditioning of the associated system equations. It is shown that the method is computationally efficient, and able to fit highly dynamic responses.

Keywords: System Identification, Rational Approximation, Time-domain modeling

1. INTRODUCTION

Accurate synthesis of compact transfer function models is crucial for reliable system-level simulations. The identification of such models from measurements or first-principle simulators is numerically not a trivial task, even for linear systems (Pintelon and Schoukens, 2001).

Recently, (Gustavsen and Semlyen, 1999) presented a frequency-domain identification method (called Vector Fitting, VF) which minimizes a weighted linear cost function, by iteratively relocating the transfer function poles using a Sanathanan-Koerner (SK) iteration (Sanathanan and Koerner, 1963; Hendrickx *et al.*, 2006). Numerical ill-conditioning is avoided by using a set of partial fraction basis functions, which are based on a well-chosen prescribed set of poles. Such rational basis functions have the advantage that an implicit weighting scheme can be applied, as described in (Deschrijver *et al.*, 2007*a*). The implicit weighting was found to give more reliable results if the prescribed poles need to be relocated over long distances.

In (Deschrijver *et al.*, 2007*b*), it was shown that the method can achieve a higher robustness if the basis functions are orthonormalized using a Gram-Schmidt procedure (called Orthonormal Vector Fitting, OVF). Using these orthonormal rational functions, the conditioning of the system equations becomes less sensitive to the initial pole specification, and accurate models can be computed in fewer iterations. This improves the efficiency of the method and reduces the overall computation time.

In this paper, a generalization of the OVF approach is presented, which allows the identification of a transfer function based on transient input-output port responses. The idea is based on a time-domain implementation of the VF technique, as proposed by (Grivet-Talocia, 2004). This paper illustrates that the advantages of orthonor-

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malization and time-domain identification can be combined. The effectiveness of the proposed technique is illustrated by an example.

2. MODEL REPRESENTATION

Traditionally, a rational transfer function $R(s)$ is defined as the quotient of two polynomials. Based on the measured or simulated spectral response $(s,H(s))$ of a physical structure, the coefficients of numerator $N(s)$ and denominator $D(s)$ are estimated, such that the least-squares distance between the model and the data is minimized (Pintelon *et al.*, 1994).

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{p=1}^{P} c_p \Phi_p(s,a)}{\tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p \Phi_p(s,a)} \tag{1}$$

In the frequency-domain OVF technique, an improved numerical robustness is obtained by expanding numerator and denominator as a linear combination of Muntz-Laguerre orthonormal rational functions $\Phi_p(s,a)$, which are based on a prescribed set of stable poles $a = \{-a_1, ..., -a_P\}$ (Heuberger *et al.*, 2005). If $-a_p$ is a real pole, then the orthonormal basis functions $\Phi_p(s,a)$ are defined as

$$\Phi_p(s,a) = \frac{\sqrt{2\,\Re e(a_p)}}{s+a_p} \left(\prod_{j=1}^{p-1} \frac{s-a_j^*}{s+a_j} \right), \tag{2}$$

and a linear combination is formed when two poles $-a_p = -a_{p+1}^*$ form a complex conjugate pair

$$\Phi_p(s,a) = \frac{\sqrt{2\,\Re e(a_p)}(s-|a_p|)}{(s+a_p)(s+a_{p+1})} \left(\prod_{j=1}^{p-1} \frac{s-a_j^*}{s+a_j} \right) \tag{3}$$

$$\Phi_{p+1}(s,a) = \frac{\sqrt{2\,\Re e(a_p)}(s+|a_p|)}{(s+a_p)(s+a_{p+1})} \left(\prod_{j=1}^{p-1} \frac{s-a_j^*}{s+a_j} \right) \tag{4}$$

It can be shown that these basis functions are orthonormal with respect to the following inner product $(1 \le m,n \le P)$

$$\langle \Phi_m(s), \Phi_n(s) \rangle_s = \frac{1}{2\pi i} \int_{i\mathbb{R}} \Phi_m(s) \Phi_n^*(s) ds \tag{5}$$

3. TRANSFER FUNCTION IDENTIFICATION

3.1 Levi's Estimator

The goal of the frequency-domain identification process, is to identify the coefficients c_p and $\tilde{c}_p$ in (1) such that the complex fitting error is minimized in a least-squares sense. Levi's linear approximation of this non-linear identification problem can be obtained by solving the following set of equations for all frequencies $\{s_k\}_{k=0}^{K_s}$ (Levi, 1959)

$$\arg\min_{c_p,\tilde{c}_p} \sum_{k=0}^{K_s} |N(s_k) - H(s_k)D(s_k)|^2 \tag{6}$$

After calculating the values of c_p and $\tilde{c}_p$, it is easy to see that equation (1) can be simplified by cancelling out the common poles a. It follows that the poles of the transfer function $R(s)$ are then actually the zeros of the denominator $D(s)$.

It is known that Levi's estimator is biased, and therefore does not guarantee convergence to the true least-squares solution. In order to relieve the unbalanced weighting, a Sanathanan-Koerner iteration can be applied (Sanathanan and Koerner, 1963). This is described in the following section.

3.2 Sanathanan-Koerner Iteration

In successive iterations $(t = 0, ..., T)$, the coefficients $c_p^{(t)}$ and $\tilde{c}_p^{(t)}$ can be updated iteratively by minimizing the following Sanathanan-Koerner cost function (Sanathanan and Koerner, 1963), provided that initially $D^{(-1)}(s) = 1$.

$$\min_{c_p^{(t)},\tilde{c}_p^{(t)}} \sum_{k=0}^{K_s} \left| \frac{N^{(t)}(s_k)}{D^{(t-1)}(s_k)} - \frac{D^{(t)}(s_k)H(s_k)}{D^{(t-1)}(s_k)} \right|^2 \tag{7}$$

$$= \min_{c_p^{(t)},\tilde{c}_p^{(t)}} \sum_{k=0}^{K_s} \left| \frac{N^{(t)}(s_k)U(s_k)}{D^{(t-1)}(s_k)} - \frac{D^{(t)}(s_k)Y(s_k)}{D^{(t-1)}(s_k)} \right|^2 \tag{8}$$

In the classical SK formulation, the coefficients $c_p^{(t)}$ and $\tilde{c}_p^{(t)}$ of $N(s)$ and $D(s)$ are estimated, provided that each equation is given an explicit frequency-dependent weighting $1/D^{(t-1)}(s)$.

The VF and OVF algorithm perform this weighting in an implicit way, by estimating the coefficients $d_p^{(t)}$ of $N^{(t)}(s)/D^{(t-1)}(s)$ and the coefficients $\tilde{d}_p^{(t)}$ of $D^{(t)}(s)/D^{(t-1)}(s)$ instead, where

$$\frac{N^{(t)}(s)}{D^{(t-1)}(s)} = \frac{1}{D^{(t-1)}(s)} \sum_{p=1}^{P} c_p^{(t)} \Phi_p(s,a) \tag{9}$$

$$= \frac{\prod_{p=1}^{P}(s+a_p) \prod_{p=1}^{P-1}(s+z_{p,n}^{(t)})}{\prod_{p=1}^{P}(s+z_{p,d}^{(t-1)}) \prod_{p=1}^{P}(s+a_p)} \tag{10}$$

$$= \sum_{p=1}^{P} d_p^{(t)} \Phi_p\left(s, z_d^{(t-1)}\right) \tag{11}$$

and

$$\frac{D^{(t)}(s)}{D^{(t-1)}(s)} = \frac{1}{D^{(t-1)}(s)} \left(\tilde{c}_0^{(t)} \sum_{p=1}^{P} \tilde{c}_p^{(t)} \Phi_p(s,a) \right) \quad (12)$$

$$\frac{\prod_{p=1}^{P}(s+a_p) \prod_{p=1}^{P}(s+z_{p,d}^{(t)})}{\prod_{p=1}^{P}(s+z_{p,d}^{(t-1)}) \prod_{p=1}^{P}(s+a_p)} \quad (13)$$

$$= \tilde{d}_0^{(t)} + \sum_{p=1}^{P} \tilde{d}_p^{(t)} \Phi_p\left(s, z_d^{(t-1)}\right) \quad (14)$$

It follows from these equations that multiplication by an explicit frequency-dependent weighting as shown in (9) and (12) is equivalent to usage of (6) if the prescribed poles of the basis functions (2)-(4) are replaced by the zeros $z_d^{(t-1)} = \left\{-z_{1,d}^{(t-1)}, ..., -z_{P,d}^{(t-1)}\right\}$ of $D^{(t)}(s)/D^{(t-1)}(s)$ in each iteration, see (11) and (14). The iterative replacement of the prescribed (or previously calculated) basis function poles is called pole-relocation. To ensure system stability, unstable poles are "flipped" into the left half of the complex plane by inverting their sign.

It was shown in (Deschrijver *et al.*, 2007*a*) that this implicit weighting approach often provides a better numerical conditioning, especially if the weighting factor $1/D^{(t-1)}(s)$ has a large dynamic variation over the frequency range of interest. Some additional improvements are obtained by scaling each column of the associated system equations to unity length. Convergence of this pole-relocation process is typically obtained in a few iterations provided that the initial set of prescribed poles a is well-chosen, i.e. as described in (Gustavsen and Semlyen, 1999).

3.3 Partial Fraction Representation

In the final iteration ($t = T$), the transfer function can be defined as the ratio of (11) and (14) such that

$$R^{(T)}(s) = \frac{\sum_{p=1}^{P} d_p^{(T)} \Phi_p\left(s, z_d^{(T-1)}\right)}{\tilde{d}_0^{(T)} + \sum_{p=1}^{P} \tilde{d}_p^{(T)} \Phi_p\left(s, z_d^{(T-1)}\right)} \quad (15)$$

It is clear that (15) can be simplified by cancelling out the relocated basis function poles $z_d^{(T-1)}$. Therefore, it follows that the poles of the transfer function are essentially the zeros of (14) at iteration step T. Based on the minimal state-space realization of $D^{(T)}(s)/D^{(T-1)}(s)$,

$$sX(s) = \mathrm{A}X(s) + \mathrm{B}U(s) \quad (16)$$
$$Y(s) = \mathrm{C}X(s) + \mathrm{D}U(s)$$

the poles $z_d^{(T)}$ of the final transfer function $R^{(T)}(s)$ can then be found by solving the eigenvalues of $\mathrm{A} - \mathrm{B}\mathrm{D}^{-1}\mathrm{C}$ (Goodwin *et al.*, 2001; Brezinski, 2002). More details about the construction of this realization are well described in (Deschrijver *et al.*, 2007*a*). Once the poles are known, the transfer function can easily be represented as a pole-residue model, by solving the residues γ_p as a linear problem.

$$\arg\min_{\gamma_p} \sum_{k=0}^{K_s} \left| \sum_{p=1}^{P} \frac{\gamma_p}{s_k + z_{p,d}^{(T)}} - H(s_k) \right|^2 \quad (17)$$

Such rational function representation can easily be realized as a SPICE equivalent circuit (Antonini, 2003).

4. TIME DOMAIN ALGORITHM

To obtain the time-domain identification algorithm, equation (8) is transformed to the time-domain using a direct application of the inverse Laplace Transformation

$$f(\mathrm{t}) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i} \int_{i\mathbb{R}} F(s)\, \mathrm{e}^{s\mathrm{t}} \mathrm{d}s. \quad (18)$$

An analogous cost function needs to be minimized, which is written in terms of an input signal $u(\mathrm{t})$ and the corresponding output signal $y(\mathrm{t})$

$$\min_{d_p^{(t)}, \tilde{d}_p^{(t)}} \sum_{k=0}^{K_\mathrm{t}} \left| \frac{N^{(t)}(\mathrm{t}_k) u(\mathrm{t}_k)}{D^{(t-1)}(\mathrm{t}_k)} - \frac{D^{(t)}(\mathrm{t}_k) y(\mathrm{t}_k)}{D^{(t-1)}(\mathrm{t}_k)} \right|^2 \quad (19)$$

provided that $u(\mathrm{t})$ and $y(\mathrm{t})$ represent the Inverse Laplace Transform of $U(s)$ and $Y(s)$ respectively. If $\phi_p(\mathrm{t}, z_d^{(t-1)})$ denotes the Inverse Laplace Transform of $\Phi_p(s, z_d^{(t-1)})$, and $f(\mathrm{t}) \star g(\mathrm{t})$ the convolution of $f(\mathrm{t})$ and $g(\mathrm{t})$, then it follows that

$$\frac{N^{(t)}(\mathrm{t})u(\mathrm{t})}{D^{(t-1)}(\mathrm{t})} = \sum_{p=1}^{P} d_p^{(t)} \left(u(\mathrm{t}) \star \phi_p\left(\mathrm{t}, z_d^{(t-1)}\right) \right)$$

$$\frac{D^{(t)}(\mathrm{t})y(\mathrm{t})}{D^{(t-1)}(\mathrm{t})} = \sum_{p=1}^{P} \tilde{d}_p^{(t)} \left(y(\mathrm{t}) \star \phi_p\left(\mathrm{t}, z_d^{(t-1)}\right) \right) + \tilde{d}_0^{(t)} y(\mathrm{t}) \quad (20)$$

In the time-domain, the basis functions $\phi_p(\mathrm{t})$ are orthonormal with respect to the time-domain inner product ($1 \leq m, n \leq P$)

$$\langle \phi_m(\mathrm{t}), \phi_n(\mathrm{t}) \rangle_\mathrm{t} = \int_0^\infty \phi_m(\mathrm{t}) \phi_n(\mathrm{t}) d\mathrm{t} \quad (21)$$

Since the Laplace transform is a unitary transformation from the time-domain to the frequency-domain, this implies that

$$\langle \mathcal{L}\phi_m(\mathrm{t}), \mathcal{L}\phi_n(\mathrm{t})\rangle_s = \langle \phi_m(\mathrm{t}), \phi_n(\mathrm{t})\rangle_{\mathrm{t}} \qquad (22)$$

and

$$\langle \mathcal{L}^{-1}\Phi_m(s), \mathcal{L}^{-1}\Phi_n(s)\rangle_{\mathrm{t}} = \langle \Phi_m(s), \Phi_n(s)\rangle_s \quad (23)$$

It is noted that the basis functions (2)-(4) are obtained by a Gram-Schmidt orthonormalization on a set of partial fractions $\{1/(s+a_p)\}_{p=1}^{P}$, provided that the poles $-a_p$ are all stable (i.e. located in the left half of the complex plane). Therefore, a Gram-Schmidt orthonormalization on the set of exponentials $\{\mathrm{e}^{-a_p \mathrm{t}}\}_{p=1}^{P}$ in the time-domain, will yield the inverse Laplace transform of the frequency-domain basis functions (2)-(4). It follows that the relevant time-domain functions $\phi_p(\mathrm{t})$ are therefore given by the inverse Laplace transform of $\Phi_p(s)$ (Titchmarsh, 1984).

The fact that no explicit expression is available for these basis functions is of no consequence, because equation (19) only needs the convolution of these functions with the input and output signals $u(\mathrm{t})$ and $y(\mathrm{t})$. To compute the filtered signals $u(\mathrm{t}) \star \phi_p(\mathrm{t})$ or $y(\mathrm{t}) \star \phi_p(\mathrm{t})$, the state space realization of the orthonormal basis functions Φ_p can be simulated with input $u(\mathrm{t})$ or $y(\mathrm{t})$, respectively.

A direct application of equation (19) to the time-domain samples $\{\mathrm{t}_k, u(\mathrm{t}_k), y(\mathrm{t}_k)\}_{k=0}^{K_{\mathrm{t}}}$ leads to a set of equations which are linear in terms of the coefficients $d_p^{(t)}$ and $\tilde{d}_p^{(t)}$. Using these coefficients and the state-space realization of the basis functions ϕ_p (which is equivalent to the realization of Φ_p), the transfer function (15) can easily be constructed. Based on this transfer function, the poles $z_d^{(T)}$ of the frequency-domain transfer function can be found by solving an eigenvalue problem, see section 3.3. Once the poles are known, the time-domain transfer function can directly be obtained by solving the coefficients γ_p as a linear problem.

$$\arg\min_{\gamma_p} \sum_{k=0}^{K_{\mathrm{t}}} \left| \sum_{p=1}^{P} \gamma_p \left(u(\mathrm{t}_k) \star \mathrm{e}^{-z_{p,d}^{(T)} \mathrm{t}_k} \right) - y(\mathrm{t}_k) \right|^2 \quad (24)$$

5. NUMERICAL EXAMPLE

The time-domain identification algorithm is applied to calculate the transfer function of a passive system, based on the transient input and output signal as partially shown in Figure 1. The system is excited with a Gaussian pulse, which is centered at t = 0.6 ns, with a width of 0.2 ns and a height of 1 in normalized units. Figure 2 shows the frequency response as a parametric curve in function of the complex frequency variable s.

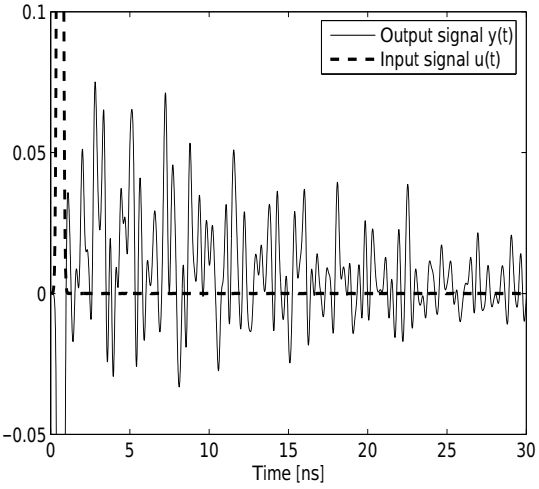

Fig. 1. Input and output signal of time-domain data over the time interval [0 ns - 30 ns]

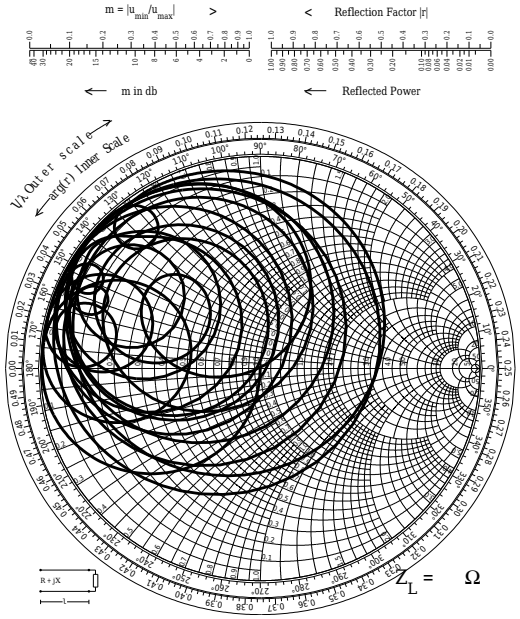

Fig. 2. Complex frequency response as a parametric curve in function of the frequency variable s (Smith Chart).

An initial set of 104 starting poles is typically chosen as proposed by (Gustavsen and Semlyen, 1999)

$$-a_p = -\alpha + \beta i, -a_{p+1} = -\alpha - \beta i$$
$$\alpha = \upsilon\beta \qquad (25)$$

where the imaginary parts β are linearly distributed over the frequency range [0 − 3 GHz], and $\upsilon = 0.01$. The parameter υ is chosen sufficiently small such that the initial poles result in a well-conditioned system matrix. The distribution of the poles over the entire frequency range reduces the probability that poles must be relocated over long distances. It is clear that other prescribed pole-location schemes are also possible,

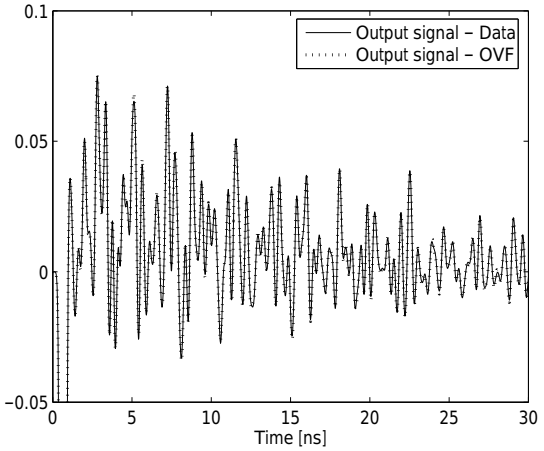

Fig. 3. Time-domain OVF approximation and reference data over the interval [0 ns - 30 ns].

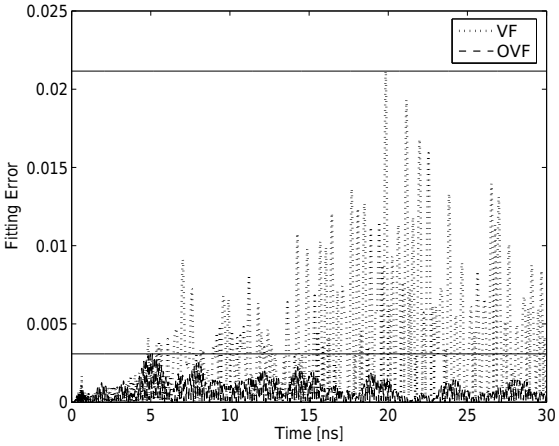

Fig. 4. Absolute fitting error of transient response over the interval [0 ns - 30 ns]. (1 iteration, $\upsilon = 0.05$)

however they often require more pole-relocation iterations.

Using the set of prescribed poles (25), the weighted linear cost function (19) is solved iteratively, and updated estimates of the model coefficients are obtained. The poles, which define the time-domain basis functions, are calculated in each iteration by solving an eigenvalue problem that is based on the estimated coefficients $\tilde{d}_p^{(t)}$. This process is repeated until the poles are converged.

In the final iteration, the time-domain basis functions are based on the converged set of relocated poles, and the overall transfer function is calculated by minimizing the cost function (24).

Due to the robustness of the orthonormal basis functions, the time-domain OVF technique is less sensitive to the initial pole specification than the time-domain VF technique presented by (Grivet-Talocia, 2004). As an example, the real part of the basis function poles is chosen to be non-negligible such that $\upsilon = 0.05$, and the algorithm is allowed to perform only 1 single iteration.

Figure 3 shows that the OVF technique provides a highly accurate approximation of the time-domain response, since there is no visible difference between the data and the transient response of the model. As can be seen from Figure 4, the maximal absolute error corresponds to 0.0032. If the same calculations are performed using the VF approach, then the maximal absolute error corresponds to 0.0212, which results in significant time-domain discrepancies.

As a means of validation, the OVF-calculated transfer function is simulated in the frequency-domain and compared to the reference spectral response. Figures 5 and 6 confirm that an overall good approximation is obtained, both in terms of the magnitude and the phase.

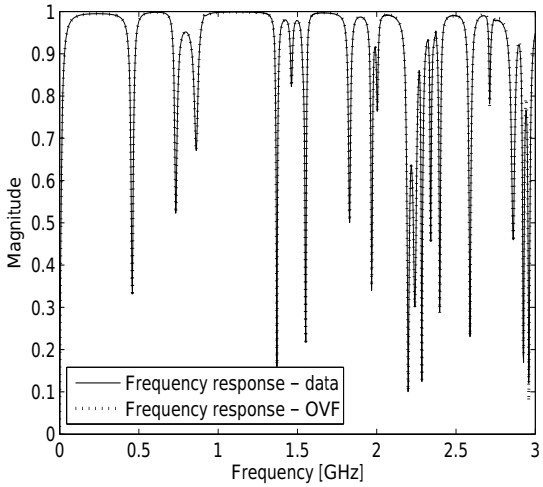

Fig. 5. Magnitude response of OVF model compared to frequency-domain reference data

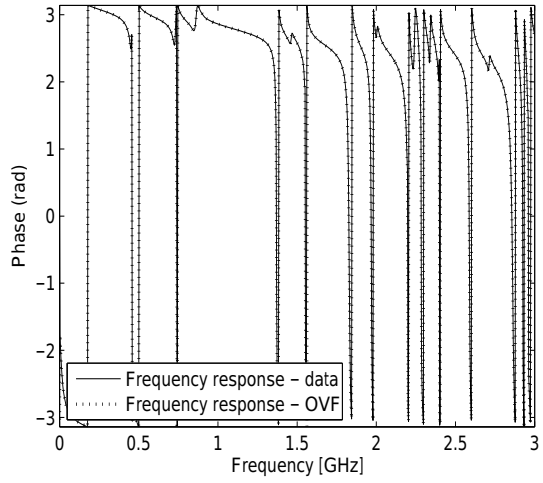

Fig. 6. Phase response of OVF model compared to frequency-domain reference data

Table 1. Maximum absolute fitting error of transient response in successive iterations ($\upsilon = 0.05$)

Iteration	VF	OVF
1	0.0212	0.0032
2	0.0032	0.0015
3	0.0032	0.0012
4	0.0017	0.0012
5	0.0012	0.0008
6	0.0010	0.0008

Table 1 illustrates that both approaches eventually converge to better, comparable results if additional iterations are performed. This results from the fact that the initial poles, (which are selected in a non-optimal way) lead to a poor numerical conditioning in the first iterations. As more iterations are performed, the poles are relocated to a better position, and the accuracy of the fitting model improves gradually. Therefore, the OVF approach is preferable, as it is numerically more robust towards to initial pole specification.

Some timing results on this example indicate that a single VF/OVF iteration takes 6.4 seconds on a P4 2.4 GHz computer with 512 MB RAM. This illustrates that the method is computationally efficient, especially since the transfer function has a relatively high order (104 poles).

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7. CONCLUSIONS

A new approach is presented for time-domain identification of transfer functions, based on transient port responses. The presented technique combines the use of a Sanathanan-Koerner iteration and an orthonormal set of basis functions to improve the numerical conditioning. It is shown that the method is numerically robust, and able to fit highly dynamic responses in an efficient way.

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