Advancements in Iterative Methods for Rational Approximation in the Frequency Domain

Dirk Deschrijver, Bjørn Gustavsen, *Senior Member, IEEE*, and Tom Dhaene, *Member, IEEE*

***Abstract*—Rational approximation of frequency-domain responses is commonly used in electromagnetic transients programs for frequency-dependent modeling of transmission lines and to some extent, network equivalents (FDNEs) and transformers. This paper analyses one of the techniques [vector fitting (VF)] within a general iterative least-squares scheme that also explains the relation with the polynomial-based Sanathanan–Koerner iteration. Two recent enhancements of the original VF formulation are described: orthonormal vector fitting (OVF) which uses orthonormal functions as basis functions instead of partial fractions, and relaxed vector fitting (RVF), which uses a relaxed least-squares normalization for the pole identification step. These approaches have been combined into a single approach: relaxed orthonormal vector fitting (ROVF). The application to FDNE identification shows that ROVF offers more robustness and better convergence than the original VF formulation. Alternative formulations using explicit weighting and total least squares are also explored.**

***Index Terms*—Macromodel, rational approximation, system identification, vector fitting (VF).**

I. Introduction

RATIONAL function identification from measured or simulated data plays an important role in the modeling of linear systems and devices. Usage of rational modeling leads to computationally highly efficient simulations as opposed to usage of direct numerical convolutions, and the inclusion in a general simulation environment is straightforward, either by recursive convolutions [1] or via an equivalent electrical circuit [2], [3]. The currently available Electromagnetic Transients Program (EMTP)-type programs for the simulation of electromagnetic transients all utilize some rational fitting method for the modeling of transmission lines and cables by the method of characteristics [1], [4], [5]. Rational fitting has also been applied for wideband modeling of frequency-dependent network equivalents (FDNEs) [2], [6]–[8] and power transformers [9]–[12].

Frequency-dependent transmission-line modeling requires fitting transfer functions for propagation and characteristic admittance over a wide frequency band. Early fitting procedures were limited to very low orders [1] until asymptotic Bode fitting with real poles was introduced in 1981 [4]. This procedure was shown to be very robust and capable of fitting modal responses using very high orders, although the accuracy was not optimal. For the fitting of nonsmooth responses that appear with FDNEs and transformers, asymptotic fitting with real poles does not apply and so Levy's linear polynomial fitting [13] was used instead [9]. Unfortunately, this approach suffers from poor numerical conditioning and unbalanced weighting, and unstable poles may also result [14]. Partitioning of the frequency band was introduced in order to overcome the ill-conditioning problem when applied to wideband data, and a Sanathanan–Koerner (SK) iteration [15] was introduced in [8] in order to relieve the biased weighting. Nonlinear optimization (for instance, Levenberg–Marquardt) has also been proposed [2] but convergence can be slow and may require a very good initial starting point.

In 1999, the vector-fitting (VF) iteration was introduced [16], [17] which proved to be a highly robust and efficient method, applicable to both smooth and resonant responses with high orders and wide frequency bands. An additional advantage is that the stability of the poles is easily enforced by a simple pole-flipping scheme. This quickly made VF become adopted in many societies of applied engineering, including power systems and microwave systems.

It has been shown [18] that the VF iteration can be viewed as a reformulation of the polynomial-based SK iteration [15]. The major difference is that VF is formulated in terms of rational functions (partial fractions) instead of the powers of s, and that the VF iteration is based on pole relocation, caused by weighting. Recently, several enhancements of VF have been proposed. The conditioning of the system equations was improved in [19] by replacing the basis functions (partial fractions) with an orthonormal set [20], thus reducing the sensitivity to the initial pole specification. The pole relocating capability was improved in [21] by relaxing the least squares nontriviality constraint in the pole identification step, thereby increasing convergence speed and accuracy of the end result. An add-on improvement by hard relocation of poles was proposed in [22].

In this paper, we present a reconciled view of VF in relation to past work and some of the recent developments. Section II explains a general iterative procedure for rational approximation with an SK cost function that can utilize different basis functions. In Section III, different basis functions are introduced: polynomials (original SK), partial fractions (VF), and orthonormal rational functions [orthonormal vector fitting (OVF)]. Pole relocation is explained as an implicit weighting scheme, and a discussion of optimal basis functions is provided. In Section IV, alternative coefficient normalizations for the VF pole identification step are presented: total least squares (TLS), and relaxed vector fitting (RVF). The significance of the applied matrix solver is explained in Section V. OVF

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D. Deschrijver and T. Dhaene are with the University of Antwerp, Antwerp 2020, Belgium (e-mail: dirk.deschrijver@ua.ac.be; tom.dhaene@ua.ac.be).

B. Gustavsen is with SINTEF Energy Research, Trondheim N-7465, Norway (e-mail: bjorn.gustavsen@sintef.no).

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and RVF are combined in Section VI into a single method [relaxed orthonormal vector fitting (ROVF)] and compared to the alternative procedures when applied to an FDNE example. In Section VII, a comparison of the time-domain simulation results is given.

II. Iterative Rational Approximation

A. Introduction

In what follows, we explain the VF iteration in the context of a general iterative technique for least-squares transfer function identification [18].

The objective is to identify the coefficients c_p and $\tilde{c}_p$ of a rational transfer function $R(s)$

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{p=1}^{P} c_p \phi_p(s)}{\tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p \phi_p(s)} \tag{1}$$

such that the difference between $R(s)$ and the data samples $(s, H(s))$ is minimized in the least-squares (LS) sense over some predefined frequency range of interest $[s_0, s_\mathrm{K}]$. We remark that the basis functions $\phi_p(s)$ were in the original SK iteration selected as monomials (powers of s) whereas VF uses rational functions. More about this is discussed in Section III.

Solving (1) corresponds to minimizing the following nonlinear cost function in terms of the system parameters [23]

$$\arg\min_{c_p, \tilde{c}_p} \sum_{k=0}^{K} \left| \frac{N(s_k)}{D(s_k)} - H(s_k) \right|^2. \tag{2}$$

One could resort to nonlinear optimization techniques to find the minimum norm solution; however, such methods are highly dependent on a very good initial starting point.

The need for nonlinear optimization is avoided by multiplying (2) with its denominator, which leads to a linear problem that minimizes Levy's cost function [13]

$$\arg\min_{c_p, \tilde{c}_p} \sum_{k=0}^{K} |N(s_k) - D(s_k)H(s_k)|^2. \tag{3}$$

Equations (2) and (3) are not equivalent, due to the introduction of an undesired weighting factor $D(s)$. This unbalanced weighting, which depends on the selected basis functions, deteriorates the quality of the fitting.

A possible option to alleviate this undesired weighting is the use of an SK iteration [15]. Here, the denominator obtained by solving (3) is used as an inverse weighting in an iterative procedure for iteration step $t = 1, \ldots, T$

$$\arg\min_{c_p^t, \tilde{c}_p^t} \sum_{k=0}^{K} \left| \frac{N^t(s_k)}{D^{t-1}(s_k)} - \frac{D^t(s_k)}{D^{t-1}(s_k)} H(s_k) \right|^2. \tag{4}$$

As the iteration (4) converges, it is assumed that $D^{t-1}(\mathrm{s})$ will approach $D^t(\mathrm{s})$, and an unbiased fitting would be achieved if $D^t(\mathrm{s}) = D^{t-1}(\mathrm{s})$ converges asymptotically to $D(\mathrm{s})$. Under these preconditions, (4) reduces exactly to (2). Although this constraint is, in practice, not always guaranteed, iteration (4) can, for sufficiently high signal-to-noise ratios and sufficiently small modeling errors, significantly improve the accuracy of the fit, since the solutions of the SK iteration relieve the bias of Levy's cost function.

B. System Equations

The weighted iterative cost function can be minimized by solving the following set of linear equations, for all complex frequencies s:

$$\frac{N^t(s)}{D^{t-1}(s)} - \frac{D^t(s)}{D^{t-1}(s)} H(s) = 0 \tag{5}$$

$$\Leftrightarrow \overbrace{w^t(s) \underbrace{\sum_{p=1}^{P} c_p^t \phi_p(s)}_{N^t(\mathrm{s})}}^{(\sigma H)^t(s)} - \overbrace{w^t(s) \underbrace{\left(\tilde{c}_0^t + \sum_{p=1}^{P} \tilde{c}_p^t \phi_p(s) \right)}_{D^t(\mathrm{s})} H(s)}^{\sigma^t(s)} = 0 \tag{6}$$

where the weighting $w^t(s)$ represents $1/D^{t-1}(s)$. Once the coefficients c_p^t and $\tilde{c}_p^t$ are unambiguously determined and converged, the transfer function $R(s)$ is well identified. ($\sigma(s)$ in (6) denotes the scaling function used in the VF paper [16]).

The weighting $w^t(s)$ is unity in the first iteration. With VF, $w^t(s)$ remains unity also in subsequent iterations as the weighting takes place implicitly by pole relocation. This is explained in more detail in Section III.

C. Numerical Aspects

Some difficulties may arise which influence the final quality of a calculated fitting model.

1) The numerical conditioning of the system equations, associated with (5) and (6), is highly dependent on the choice of basis functions and the behavior of the weight function $w^t(s)$, see Section III.
2) An appropriate normalization must be imposed on the coefficients of the transfer function in order to avoid the trivial null solution. The choice of normalization can strongly influence the convergence of (4): see Section IV.
3) A reliable matrix solver must be used: see Section V.

The first issue was addressed in the VF method [16], by introducing partial fractions as basis functions and implicit weighting by pole relocation, which greatly improved the conditioning as compared to the usage of monomials and explicit weighting in the SK method [15]. In [19], the OVF method was proposed to improve the conditioning even further by applying a Gram–Schmidt orthonormalization on the partial fractions. On the other hand, the RVF method [21] was introduced to improve the coefficient normalization. In this paper, it is proposed to combine the benefits of both methods, which leads to the ROVF.

III. Basis Functions and Pole Relocation

A. Iteration Using Polynomial Basis (SK)

In the original SK algorithm [15], the formulation of (6) is based on a power series basis for numerator $N^t(s)$ and denominator $D^t(s)$. Unfortunately, this approach suffers from poor numerical accuracy when calculating high-order approximations over a wide frequency band, as the polynomial basis functions s^n lead to the solution of a poorly conditioned system. The SK weighting $w^t(s)$ can sometimes contribute to this problem. In

addition, some poles often result in the right-half plane, leading to unstable time-domain simulations.

The numerical conditioning of the equations can be improved by using orthonormal polynomials [24]. However, the orthonormalization of the polynomials as well as the evaluation according to the three-term recurrence relation can be time consuming, and the calculation of the recurrence coefficients can be ill-conditioned.

B. Iteration Using Partial Fractions Basis (VF)

The VF algorithm [16] resolves these issues by using a common set of partial fractions as basis functions for the numerator and denominator

$$\phi_p(s,a) = \frac{1}{s-a_p}. \tag{7}$$

Since the numerator $N(s)$ and denominator $D(s)$ share a common set of poles, the transfer function is reduced to a linear combination of partial fractions after simplification. It was found that this leads to a significant improvement in numerical conditioning when the initial poles are well chosen.

Direct application of (6) results in an iterative procedure with a nonzero $w^t(s)$. However, in the VF formulations, the weighting takes place implicitly by pole relocation. This results in the fact that the basis functions are updated while $w^t(s)$ remains unity. The new poles are identified in each iteration by solving an eigenvalue problem of the form [25, p. 612]

$$z_d^t = \mathrm{eig}\left(A^t - b\left(\tilde{c}_0^t\right)^{-1} c^t\right) \tag{8}$$

where A^t, b, $\tilde{c}_0^t$, and c^t are obtained from the state space realization of $\sigma^t(s)$. These new poles z_d^t replace the previously estimated poles, and the iteration process is repeated until convergence, which implies that $\sigma(s)$ in (6) tends to unity at all frequencies.

Although this pole relocating procedure is mathematically equivalent to the usage of (6) with an explicit weighting $w^t(s)$ and no pole relocation, the implicit weighting of VF leads to better conditioning and more accurate results as will be demonstrated in Section VI-C. As the poles are directly available, any unstable poles may be flipped into the left-half plane during each iteration; hence, enforcing the stability of the poles. A further improvement of the conditioning is achieved by normalizing the columns to unit length.

Based on the identified final poles, the residues are obtained by solving the following linear problem:

$$\arg\min_{c_p^t} \left| H(s) - \sum_{p=1}^{P} c_p^t \phi_p\left(s, z_d^t\right) \right|^2 \tag{9}$$

which leads to a rational approximation in partial fraction form.

For the first iteration, the initial poles a_p should be chosen to occur in complex conjugate pairs with weak attenuation and be distributed over the frequency range of interest, for example

$$a_p = -\alpha_p + j\beta_p, a_{p+1} = -\alpha_p - j\beta_p \tag{10}$$

where

$$\{\alpha_p\} = v \cdot \{\beta_p\}. \tag{11}$$

The parameter v must be chosen such that the initial poles result in a well-conditioned system matrix. Typically, a value of 0.01 is used, giving well-conditioned basis functions both with linearly spaced and logarithmically spaced samples and poles. The distribution of the poles over the entire frequency range reduces the probability that poles must be relocated over long distances, thus increasing convergence speed. When fitting fundamentally smooth functions as in transmission-line modeling by the method of characteristics, real initial poles that are logarithmically distributed over the frequency range of interest may also be used.

C. Iteration Using Orthonormal Rational Basis (OVF)

In [26], it was argued that this choice of partial basis functions (7) is not optimal. A set of orthonormal rational functions $\varphi_p(s)$ is proposed [19] that is orthonormal with respect to the following inner product:

$$\langle \varphi_m(s), \varphi_n(s) \rangle = \frac{1}{2\pi j} \int_{j\Re} \varphi_m(s) \varphi_n^*(s) ds = \delta_{mn} \tag{12}$$

for $1 \leq m, n \leq P$. Based on a Gram–Schmidt orthonormalization of the partial fractions (7), the following closed-form expression is obtained:

$$\varphi_p(s,a) = \frac{\kappa_p \sqrt{-2\Re e(a_p)}}{s-a_p} \left(\prod_{i=1}^{p-1} \frac{s+a_i^*}{s-a_i} \right) \tag{13}$$

provided that the poles are stable. κ_p represents an arbitrary unimodular complex number, which is set to 1 in practice. Note that these basis functions are essentially a linear combination of the partial fractions which are used in the VF method. Real valuedness is enforced by forming a linear combination of two basis functions, and the calculation of the zeroes of $D(\mathrm{s})$ is generalized. The iteration takes place by pole relocation instead of explicit weighting. More implementation details are described in the OVF paper [19].

Since the orthonormalization of the basis functions is performed analytically instead of numerically, no additional computational cost is introduced by using this basis.

As was shown in [19], this approach can significantly improve the numerical conditioning of the system equations, especially if the real part of the initial poles is non-negligible compared to the poles of the transfer function. This can reduce the number of required iterations and, hence, the total computation time of the fitting process. When sufficient iterations are performed, the VF method and the OVF method generally converge to comparable results. A numerical interpretation of the orthonormal basis functions versus partial fractions is shown in the Appendix.

D. Optimality of the Basis Functions

Even though the orthonormal rational functions as proposed in (13) improve the numerical conditioning of the system equations, optimal conditioning is not yet attained since the basis

functions are orthonormalized with respect to the wrong inner product. This is discussed in the following.

In [27], the three-term recurrence for orthonormal polynomials is generalized for rational basis functions. To identify the transfer function, the numerator and denominator can be expanded in a different set of basis functions $\Psi(\mathrm{s})$ and $\Phi(\mathrm{s})$, respectively. Both bases are obtained, for example, by a Gram–Schmidt orthonormalization of the partial fractions (7), with respect to the following discrete inner product:

$$\langle \Upsilon_m(s), \Upsilon_n(s) \rangle = \sum_k \omega_k \Upsilon_m(s_k) \left(\omega_k \Upsilon_n(s_k)\right)^* = \delta_{mn}. \tag{14}$$

The weighting factor ω_k is fixed to 1 for the numerator basis functions ($\Upsilon = \Psi$), and fixed to $H(s_k)$ for the denominator basis functions ($\Upsilon = \Phi$). If U and V are defined as

$$U_r = [\psi_r(s_0) \dots \psi_r(s_K) \psi_r(s_0^*) \dots \psi_r(s_K^*)]^T \tag{15}$$

$$V_r = [H(s_0)\Phi_r(s_0) \dots H(s_K)\Phi_r(s_K) H^*(s_0)\Phi_r(s_0^*) \dots H^*(s_K)\Phi_r(s_K^*)]^T \tag{16}$$

$$U = [U_1 \dots U_P] \text{ and } V = [V_0 \dots V_P]. \tag{17}$$

Then, the normal equations $(A^{*^T} A)x = A^{*^T} b$ are given as

$$\begin{pmatrix} Y & X \\ X^T & Z \end{pmatrix} x = \begin{pmatrix} G \\ F \end{pmatrix} \tag{18}$$

with $Y = (U^*)^T(U)$, $Z = (V^*)^T(V)$, $X = -\Re e((U^*)^T(V))$, $G = \Re e((U^*)^T(V_0))$, and $F = \Re e((V^*)^T(V_0))$. Due to the orthonormality of the basis functions $Y = Z = I$, where I represents the identity matrix of appropriate dimension, and $F = 0$.

It can be proven that any basis transformation from this dual basis approach to an arbitrary rational basis that spans the same space will result in inferior numerical conditioning of the normal equations. The proof for polynomial bases is given in [28], and extends in a natural way to rational bases. Since the condition number $\kappa(A^{*T}A) = (\kappa(A))^2$, the conditioning of A is optimal. Unfortunately, the gain in numerical accuracy is now lost in the calculation of the recurrence coefficients. This approach makes it much harder to calculate the poles and zeroes of the transfer function accurately, which is a necessary requirement if this basis is to be combined with the implicit iterative procedure (6). Also, the orthonormalization of these basis functions is numerically not a trivial task, and the computational complexity is much higher than that of the VF or OVF algorithm. In this perspective, the basis functions (13) are preferable.

IV. Coefficient Normalization

A. Relaxed Vector Fitting (RVF)

Experience with the original VF algorithm has shown that its convergence properties become severely impaired if the response to be fitted is contaminated with noise [22]. It was shown in [21] that this problem is also related to the adopted LS normalization where $(\tilde{c}_0)$ is set to be equal to 1. With this normalization, the required pole relocation may result in an increase of the LS error of (6). Since the LS solver gives a minimum error, the poles are relocated in small steps and the convergence may even stall. Another problem is that the LS solution tends to produce a σ with a small magnitude in the fitting range since this scales down the fitting error and, thus, produces a smaller LS error [31]. The latter implies that the solution of (6) is biased and that the relocated poles will not minimize the original problem (2). For instance, when fitting smooth functions, one often finds that the poles tend to shift too much toward low frequencies.

In [21], a modification to the VF algorithm was introduced that alleviates the previously mentioned difficulties by improving the normalization of the transfer function coefficients and the linearization of the SK iteration at the same time. This is achieved by introducing the more relaxed nontriviality condition (19) as an additional row in the system matrix. The constraint (19) simply imposes that the sum of the $\sigma(s)$ samples approaches a nonzero value, without fixing any of its coefficients

$$\Re \mathrm{e} \left\{ \sum_{k=0}^{K} \left(\sum_{p=1}^{P} \tilde{c}_p \phi_p(s_k) + \tilde{c}_0 \right) \right\} = K + 1. \tag{19}$$

This equation is given an LS weighting in relation to the size of $H(s)$ by

$$\text{weight} = \|H(s)\| / (K+1). \tag{20}$$

It is noted that $\sigma(s)$ will still approach unity for all frequencies as the iteration converges.

Use of the constraint (19) alleviates the downscaling phenomenon since the sum of the $\sigma(s)$ samples is fixed. This can be understood by considering that a reduction in LS error achieved by downscaling $\sigma(s)$ in some frequency range will be offset by a magnification of the LS error at other frequencies. At the same time, the freed variable $\tilde{c}_0$ improves the pole relocating capability as the error magnification caused by a given pole relocation is reduced.

B. Total Least Squares (TLS)

An alternative way of making $\tilde{c}_0$ a free variable and, thus, improving convergence and conditioning is to fix the Euclidean norm of the parameter vector equal to 1. This gives rise to a TLS problem [30]. Equation (6) reduces to a linear system of least-squares equations $Ax = b$, in terms of the unknowns $x = \{c_1, \dots, c_P, \tilde{c}_0, \dots, \tilde{c}_P\}$. Based on a QR decomposition of $A = QR$, a smaller system of equations is obtained

$$Q^T A x = \begin{pmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,2P+1} \\ 0 & r_{2,2} & \dots & r_{2,2P+1} \\ 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & r_{2P+1,2P+1} \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} x = Q^T b \tag{21}$$

which reduces to the following form:

$$\begin{pmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,2P+1} \\ 0 & r_{2,2} & \dots & r_{2,2P+1} \\ 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & r_{2P+1,2P+1} \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}. \tag{22}$$

This minimization problem is of the form $\|Bx\|_2 = \min$, subject to the normalization constraint $\|x\| = 1$. The solution

of this minimum corresponds to the smallest singular value of the matrix B, and the solution is given by the corresponding singular vector [30].

Since the constant term of the denominator is now variable, the zeros are calculated by solving the eigenvalue problem (8).

There are, however, problems with nonmonotonous convergence when using TLS, as will be demonstrated by the numerical examples in Section VI-E. Also, the TLS formulation does not address the downscaling problem described in Section IV-A. The use of TLS requires the calculation of an SVD, which is computationally expensive.

V. Matrix Solver

SK, VF, OVF, and RVF all lead to the solution of a linear problem of the form

$$Ax = b \tag{23}$$

where the columns of A are formed by the basis functions (one row per frequency). If X is defined as

$$X = \left[\phi_1\left(s, z_d^{t-1}\right) \quad \ldots \quad \phi_P\left(s, z_d^{t-1}\right)\right] \tag{24}$$

then considering the case with $\tilde{c}_0 = 1$, we get

$$A = [X \quad -HX] \text{ and } b = H. \tag{25}$$

Despite the fact that the left and right blocks of A can each be made well conditioned as was explained in Section III-D, the full matrix can still become ill-conditioned. This type of ill-conditioning will, however, not always result in a poor fitting model provided that a solver is used which can handle poorly conditioned systems. This can be achieved using SVD or QR decomposition with column pivoting [32]. The latter approach is used when applying the $\backslash$ operator in Matlab.

VI. Example: FDNE Identification

A. Approach

In this example, the FDNE identification of a power distribution system is considered. The system has two three-phase buses as terminals (A, B), and is shown in Fig. 1. The 6×6 admittance matrix Y is calculated with respect to these terminals in the frequency range 10 Hz–100 kHz, see magnitude functions in Fig. 2. All lines and cables are modeled in the phase domain, taking into account frequency-dependent effects in conductors and ground.

In the following, the accuracy of multiple 50th-order common-pole rational approximations is compared, each calculated with a different version of the VF algorithm as listed in Table I. Stable poles are enforced in each iteration by pole flipping.

The calculated results are based on linearly spaced and complex conjugate initial poles with a fixed attenuation factor

$$\{\alpha_p\} = \upsilon \cdot \beta_{\max} \tag{26}$$

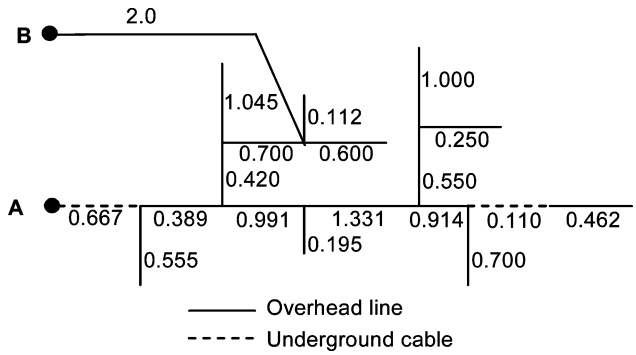

Fig. 1. Power system distribution system (lengths in kilometers).

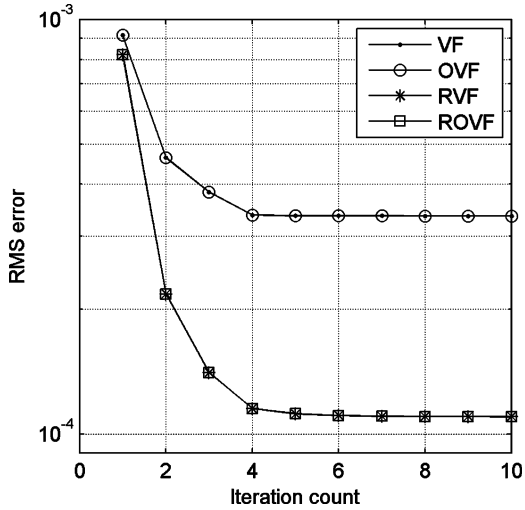

Fig. 2. Weakly attenuated initial poles ($\nu = 0.01$).

TABLE I
Algorithms

VF	Classical Vector Fitting with partial fraction basis
OVF	VF with orthonormal basis functions
RVF	VF with relaxation of non-triviality constraint
ROVF	VF with relaxation and orthonormalization

with $\nu = 0.01$ or $\nu = 1.0$. This choice is also allowed [instead of (11)], since both frequency samples and poles are linearly distributed over the frequency range. Note that the basis functions become poorly observable with $\nu = 1.0$.

B. Result With Alternative Initial Pole Sets

Fig. 2 shows the evolution of the root mean square (rms) fitting error as a function of iteration count when the initial poles have small real parts ($\nu = 0.01$). It can be seen that VF and OVF give comparable results since the initial poles lead to a set of well-observable basis functions in this case. RVF and ROVF give a smaller fitting error due to the relaxation of the nontriviality constraint. Fig. 3 shows the final fitting result by RVF/ROVF.

The calculations are repeated, now using initial poles with strong attenuation ($\nu = 1.0$). The result in Fig. 4 shows that

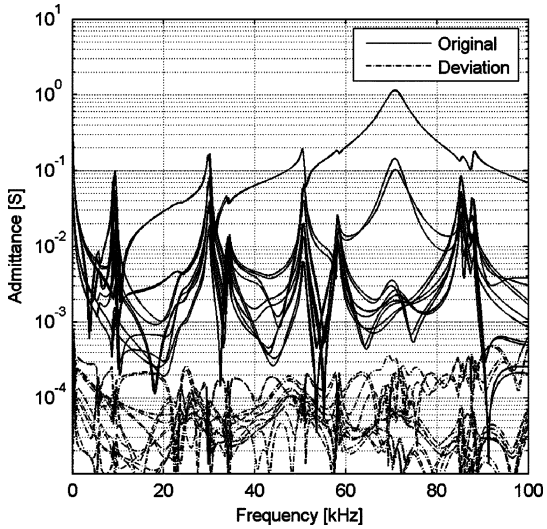

Fig. 3. Resulting rational approximation (RVF/ROVF). Ten iterations.

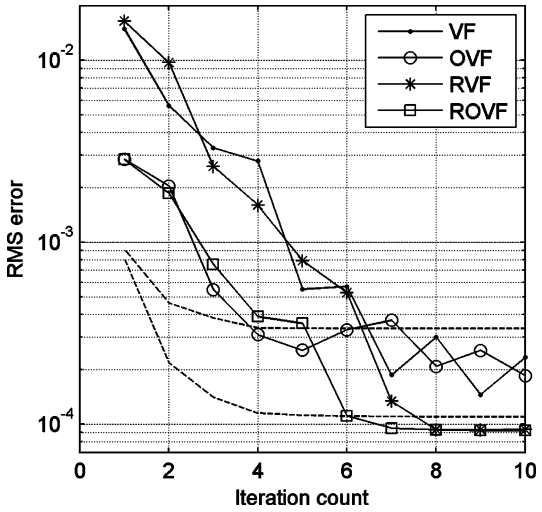

Fig. 4. Strongly attenuated initial poles ($\nu = 1$).

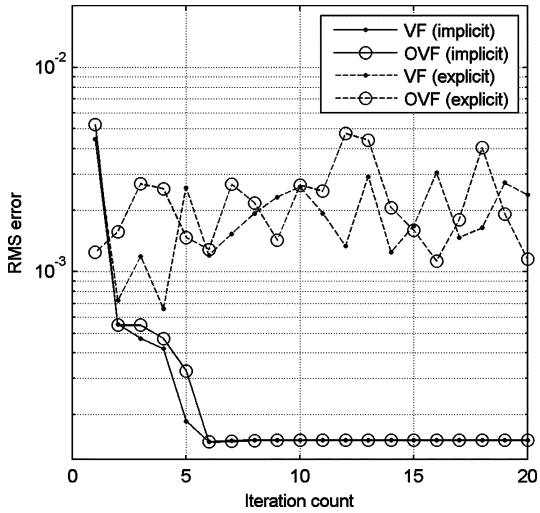

Fig. 5. Explicit weighting versus implicit weighting (pole relocation).

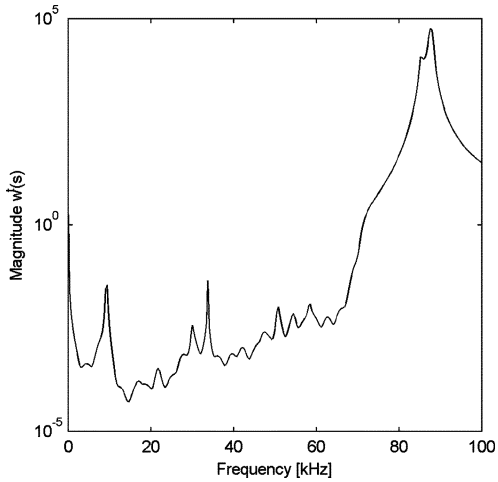

Fig. 6. Magnitude of explicit weighting factor $w^t(s)$ (final VF iteration step).

orthonormalization leads to significantly faster convergence of the iterations. This occurs because the initial basis functions of VF are now very smooth and, thus, poorly observable, and so the first iteration produces inaccurate poles. But the procedure still converges since the new poles are less attenuated than the previous poles and so the conditioning improves with iteration count.

For comparison, the dashed lines in Fig. 4 show the result that was obtained with the previous initial pole set (Fig. 2). It is seen that the use of weakly attenuated poles leads to faster convergence with all approaches and should thus be the preferred choice. It is also seen that the final rms error depends on the initial pole set. This confirms the assertion in Section IV-A that the iterative procedure (6) will, in general, not lead to the true optimal solution.

C. Result of Implicit Versus Explicit Weighting

As was explained in Section III-B, the iteration (6) is with the VF approach formulated in terms of pole relocation and no explicit weighting $w^t(s) = 1$. This means that the weighting occurs implicitly. In order to investigate the significance of pole relocation, an alternative VF formulation with fixed poles and explicit weighting was implemented. Usage of explicit weighting was found to be inferior in situations where the initial poles need to be relocated over long distances. To demonstrate this phenomenon, the initial poles are in Fig. 5 equally distributed over the lower 70% of the frequency range of interest. This causes the weighting function $w^t(s)$ to have a large magnitude at high frequencies and a small magnitude at low frequencies, as shown in Fig. 6. Due to the large dynamic variation of the weighting function, the numerical conditioning of the system equations breaks down, and a poor fitting model is obtained. Note that the accuracy does not improve during iterations as opposed to the formulation with pole relocation.

D. Result With TLS

In order to illustrate the usefulness of the relaxation, we compare the obtained result with that by an alternative relaxation scheme: TLS (Section IV-B). The latter method fixes the norm of the parameter vector to unity, resulting in a different coefficient normalization. It is seen in Fig. 7 that although the TLS

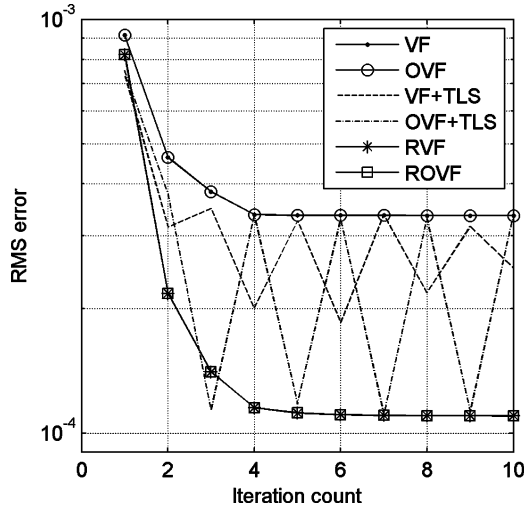

Fig. 7. Total least-squares algorithm ($\nu = 0.01$).

TABLE II
COMPUTATION TIME (in seconds)

Method	Fitting single element		Fitting 21 elements	
	N=10	N=50	N=10	N=50
VF	0.043	0.16	0.44	3.4
RVF	0.040	0.16	0.45	3.4
ROVF	0.076	0.22	0.49	3.5

algorithm can give a more accurate result than the classical VF and OVF approach, the error tends to oscillate strongly. Its accuracy is also inferior to that of RVF and ROVF.

E. Computational Efficiency

Table II compares the computation time obtained with VF, RVF, and ROVF. The result is shown for a Matlab implementation using sparsity and normal equations. Since Matlab is a non-compiled language, the timing results are sensitive to the actual implementation. The responses contain 300 frequency samples.

VII. TIME-DOMAIN SIMULATION

The following compares simulation results by the FDNE model when identified using alternative VF formulations. This is achieved by including the rational model in the PSCAD/EMTDC environment with a user-defined subroutine [35]. Unfortunately, most simulations were unstable due to passivity violations. In order to compare simulation results by the different VF alternatives, we could not remove the passivity violations as the resulting perturbation would mask the difference between the alternatives. Therefore, we only show results with the rational model used as a pure transfer function model, see Fig. 8. In this example, bus A is energized by a three-phase voltage source while the far-end terminals are short circuited to ground. The voltages at the terminals are taken as input quantities while the currents at the terminals are calculated using convolutions. The voltage is ramped up from zero in 10 μs to suppress high-frequency transients above 100 kHz, which is the upper frequency limit in the rational approximation.

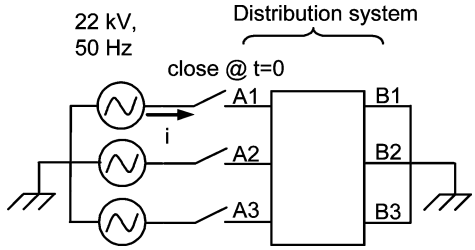

Fig. 8. Energization of the network in Fig. 1.

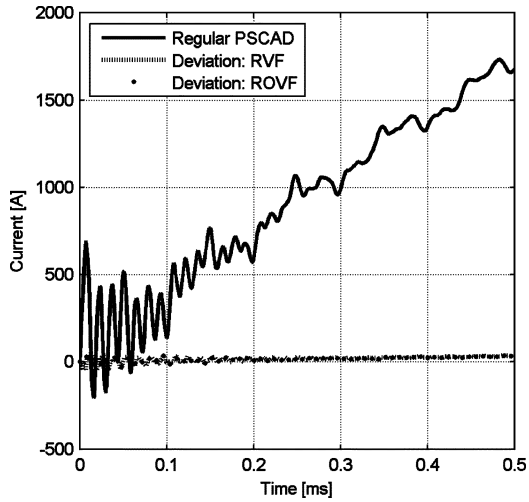

Fig. 9. Current in terminal 1. Simulation with $\nu = 0.01$ and two iterations.

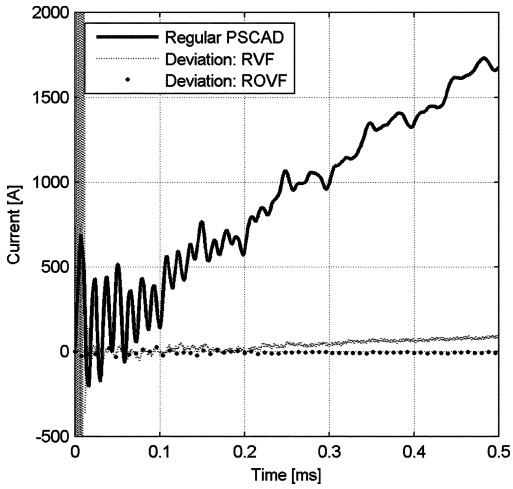

Fig. 10. Current in terminal 1. Simulation with $\nu = 1$ and six iterations.

Fig. 9 compares the simulated current flowing into terminal A1 when using RVF and ROVF, with $\nu = 0.01$ (weakly attenuated poles) and two iterations. As a reference, the result by a regular PSCAD simulation is shown, where each line stub is modeled using the phase-domain line model. The RVF and ROVF results are virtually identical and very close to the regular PSCAD solution.

In Fig. 10, the same result is shown when using $\nu = 1$ (strongly attenuated initial poles) and six iterations. Here, the ROVF result still agrees well with the PSCAD solution, while the RVF result differs significantly.

VIII. Discussion

In Section III, a generalized SK iteration was introduced by the introduction of alternative basis functions. The VF iteration was shown to be an implementation using rational basis functions (partial fractions) instead of a polynomial basis, and implicit weighting (pole relocation) instead of explicit weighting. In Section VI-C, it was shown that the use of pole relocation is essential for obtaining robust procedure.

In Section VI, various versions of VF were applied to the rational fitting a six-port FDNE. The calculated results in Fig. 5 showed that the use of orthonormal basis functions (OVF), instead of partial fractions (VF) give substantially faster convergence when the initial poles are chosen with relatively large real parts ($\nu = 1$). This is caused by the fact that the partial fraction basis functions become poorly observable. The two approaches do, however, converge to similar results since the attenuation factor of the relocated poles is typically reduced as the iteration goes on and, thus, the conditioning improves. This improvement in conditioning is a direct consequence of the implicit weighting (pole relocation) used in VF. The use of small real parts ($\nu = 0.01$) remains, however, as the preferred choice with all VF versions since this gives the fastest convergence.

One might expect that when fitting a smooth function (for instance, transmission line modes), OVF would always outperform VF since the poles will necessarily converge toward real poles and, thus, result in poorly observable basis functions. It has, however, been found that the performance of both approaches remains similar for smooth functions, even when using real initial poles [31]. This surprising result is related to the fact that (24) becomes ill-conditioned also with orthonormal basis functions. One might say that the observability problem now lies in the (smooth) frequency response itself and not in the basis functions.

In Section VI, it was also shown that normalization of (6) by the relaxation of (19) and (20) (RVF/ROVF) leads to faster convergence and a more accurate end result than the nontriviality constraint of VF/OVF which fixes the constant term in $\sigma(s)$ to unity ($\tilde{c}_0 = 1$). The improvement is particularly significant when fitting noisy responses [21] (not shown in this paper). An alternative normalization using TLS gave a result with a strongly oscillating rms error, lying somewhere between that of VF and RVF.

In Section VII, a time-domain simulation was shown with the FDNE interfaced to the PSCAD simulation program. Comparing the RVF and ROVF simulation results with that of a regular PSCAD simulation verified that with $\nu = 1$ and six iterations, ROVF produces a more accurate result than RVF. However, with $\nu = 0.01$, both approaches produced a highly accurate result with only two iterations.

We therefore consider ROVF to be the best approach since it combines the advantages of both OVF and RVF. As was shown in Section VI-E, the additional computational effort by orthonormalization and relaxation is negligible for the fitting of responses with high orders and several elements.

This paper has shown results that were obtained by various formulations of VF. We have also tried polynomial SK fitting (power series basis) with the scaling of frequency [33] and columns, but the numerical results were unacceptable due to ill-conditioning. The usage of polynomial fitting requires partitioning of the frequency band [8] or the use of orthonormal polynomials [28].

The use of Whitfield's cost function, as an alternative to the SK cost function, was investigated in [34]. Although the use of Whitfield iterations should in theory converge closer toward the true optimum solution, the method was found to be prone to divergence.

IX. Conclusion

This paper has investigated the iterative technique for rational approximation known as VF in terms of past work and recent enhancements. The main conclusions are as follows.

1) The VF iteration can be viewed as a reformulation of the SK iteration with rational basis functions instead of polynomials, where the weighting between successive iterations occurs implicitly by pole relocation. Without pole relocation, the numerical performance would be impaired.
2) Usage of orthonormal basis functions (OVF) instead of partial fractions (VF), leads to better numerical conditioning and faster convergence when the initial poles have been specified with large real parts.
3) Normalization by relaxation (RVF) leads to faster convergence and a better end result than VF. Since orthonormalization and relaxation can be performed with negligible computational cost, these two enhancements should always be used, leading to ROVF.
4) Normalization by TLS has been investigated but could not rival the performance of ROVF.

Appendix
Conditioning

It has been observed in several publications (e.g., in [19] that the orthonormal basis functions (13) lead to better conditioned system equations if the real part of the initial poles is chosen to be non-negligible).

This can be explained by comparing the structure of the basis functions. Note that orthonormal basis functions (13) of OVF are essentially the partial fractions of VF, multiplied with some all-pass functions and a normalization constant.

First, the effect of the all-pass functions is investigated. Consider, for example, a set of ten basis functions, which are based on the following set of real and stable poles $a = \{-1, -2, \ldots, -10\}$.

Fig. 11 shows the phase of the partial fraction basis, while Fig. 12 shows the phase of the unnormalized, orthogonal basis functions over the frequency range $[-20j, 20j]$.

Clearly, the multiplication with all-pass functions does not influence the magnitude of the basis functions; however, the phase of the unnormalized, orthogonal basis functions shows more variation with an increase in order. Also, the phase variation now ranges nicely from $-\pi$ to π. This increase of dynamic behavior improves the observability of the complex basis functions, and leads to an improvement in numerical conditioning.

Similar results can be obtained for complex conjugate initial poles. Note, however, that the influence of the all-pass functions becomes less pronounced as the real part of the poles decreases.

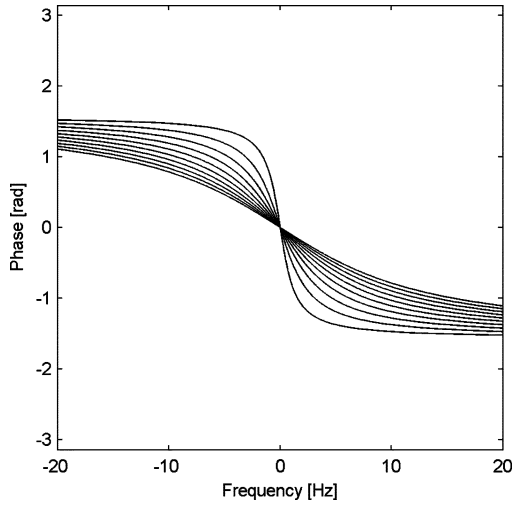

Fig. 11. Phase of partial fraction basis functions.

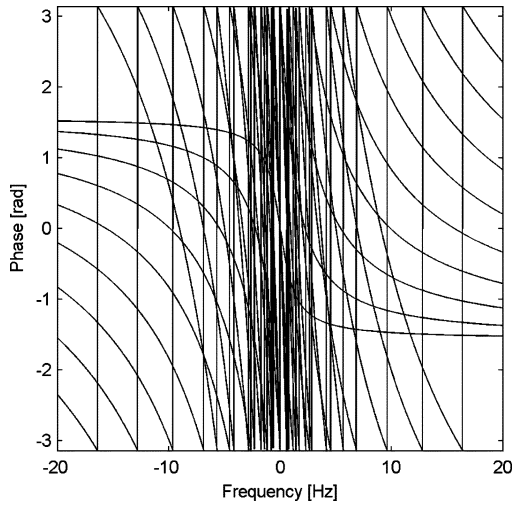

Fig. 12. Phase of unnormalized orthogonal basis functions.

Assume, for sake of simplicity, that a pole $a_{\rm p} = 0 + j\omega_{\rm p}$ has a zero-real part, then the all pass function

$$\frac{s + a_p^*}{s - a_p} = \frac{j\omega - j\omega_p}{j\omega - j\omega_p} = 1 \tag{27}$$

is simply reduced to a multiplication by 1. Consequently, the unnormalized orthogonal basis is almost reduced to the partial fraction basis if the real part of the poles is chosen to be small enough. This explains why the difference in numerical conditioning between VF and OVF becomes smaller if the real part of the initial poles is chosen small.

Second, the influence of the normalization constant is investigated. The normalization of the unnormalized orthogonal basis functions by $\sqrt{-2\Re e(a_p)}$ can basically be considered as a scaling of the columns of the system equation matrix (24). It was already observed in [29] that this can significantly improve the condition number of the system matrix, and so can lead to more accurate results.

This concept was already implemented in the VF technique by scaling the columns of the system matrix to unity length. It should be noted that this scaling is not redundant for the OVF method, as the inner product (16) does not incorporate the function values, such as the optimal inner product (18). Also, the scaling to unity length may be necessary if additional polynomial terms $(1, s, s^2, \ldots)$ are added to the numerator, to vary the order of the transfer function for nonstrict proper systems.

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Dirk Deschrijver was born in Tielt, Belgium, on September 26, 1981. He received the M.S. degree in computer science from the University of Antwerp, Antwerp, Belgium, in 2003.

Currently, he is with the Computer Modeling and Simulation (COMS) Group, University of Antwerp, supported by a research project of the Fund for Scientific Research Flanders (FWO-Vlaanderen). His research interests include rational least-squares approximation, orthonormal rational functions, system identification, and macromodeling.

Mr. Deschrijver was a Marie Curie Fellow at the Eindhoven University of Technology, Eindhoven, The Netherlands.

Bjørn Gustavsen (M'94–SM'03) was born in Harstad, Norway, in 1965. He received the M.Sc. and Dr.ing. degrees from the Norwegian Institute of Technology (NTH), Trondheim, Norway, in 1989 and 1993, respectively.

Since 1994, he has been with SINTEF Energy Research, Trondheim. His interests include simulation of electromagnetic transients and modeling of frequency-dependent effects. He was a Visiting Researcher with the University of Toronto, Toronto, ON, Canada, in 1996, and with the Manitoba HVDC Research Centre, Winnipeg, MB, Canada, in 1998.

Dr. Gustavsen was a Marie Curie Fellow at the University of Stuttgart, Stuttgart, Germany, from 2001 to 2002.

Tom Dhaene (M'02) was born in Deinze, Belgium, on June 25, 1966. He received the Ph.D. degree in electrotechnical engineering from the University of Ghent, Ghent, Belgium, in 1993.

From 1989 to 1993, he was Research Assistant in the Department of Information Technology, the University of Ghent, where his research focused on different aspects of full-wave electromagnetic-circuit modeling, transient simulation, and time-domain characterization of high-frequency and high-speed interconnections. In 1993, he joined the EDA company Alphabit (now part of Agilent). He was one of the key developers of the planar EM simulator ADS Momentum, and he is the principal developer of the multivariate EM-based adaptive metamodeling tool ADS Model Composer. Since 2000, he has been a Professor in the Department of Mathematics and Computer Science at the University of Antwerp, Antwerp, Belgium, As author or coauthor, he has contributed to more than 100 peer-reviewed papers and abstracts in international conference proceedings, journals, and books. He is the holder of 2 U.S. patents.