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Iterative Rational Least-Squares Method for Efficient Transfer Function Synthesis

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Abstract

This paper describes an iterative rational least-squares method, which is used for accurate transfer function synthesis of frequency-domain continuous-time systems. The identification method starts from an initial set of prescribed poles, and relocates them using a Sanathanan-Koerner iteration in order to minimize the global fitting error. Orthonormal rational functions are used to improve the numerical conditioning of the system equations. The method is computationally very efficient, and the calculated transfer function is very lenient towards accurate extraction of poles and zeros.

1 Introduction

Frequency domain least-squares identification methods are known to suffer poor numerical conditioning if the frequency range is rather broad, or when the transfer function requires a large amount of poles. Numerous techniques are proposed in literature to relieve some of these issues.

Richardson and Formenti [1] proposed the use of Forsythe polynomials which are orthonormal with respect to a discrete inner product, defined by the normal equations of the estimator. This implies that a different set of basis functions is used for numerator and denominator. Rolain et al. [2] have shown that the Forsythe polynomial basis is optimal in a sense that there doesn't exist any other polynomial basis resulting in a better conditioned form.

In [3], an orthonormal rational basis is proposed, which generalizes the 3-term recurrence relation for orthonormal polynomials. This approach minimizes the condition number of the system equations, by choosing an inner product which matches that of the cost function. To remove the non-linearity of the approximation problem, it is assumed that a-priori knowledge about the poles is available.

In this paper, an identification algorithm is used which minimizes a weighted linear cost function, by iteratively relocating a set of prescribed poles using a Sanathanan-Koerner iteration [4]. The numerical conditioning of the system equations is improved by using Muntz-Laguerre orthonormal rational functions [5]. Even though this choice of basis doesn't lead to optimal conditioning, it should be noted that the quality of a fit isn't solely dependent on its condition number. In fact, the model accuracy also depends the accuracy of the computation of the basis functions, the calculation of the recurrence coefficients, the matrix solver, and the extraction of poles and zeros from the transfer function. In this setting, the Muntz-Laguerre basis is much easier to handle, without sacrificing significant loss in accuracy [6].

2 Non-linearity of the problem

A rational transfer function $R(s)$ is defined as a quotient of two polynomials $N(s)$ and $D(s)$. In this paper, the numerator and denominator of the transfer function are represented as a linear combination of P orthonormal rational functions $\phi_p(s)$.

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{p=1}^{P} c_p \phi_p(s)}{\tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p \phi_p(s)} \quad s = i2\pi f \tag{1}$$

The denominator has an additional basis function which equals the constant value 1, and the coefficients c_p and $\tilde{c}_p$ represent the model coefficients. Given a set of Laplace data samples $(s_k, H(s_k))$, the transfer function should

match the data in a least squares sense, such that $R(s_k) \simeq H(s_k)$, for $k = 0, ..., K$. These data samples can be deterministic, e.g. obtained from circuit simulations, or they can be noisy, e.g. obtained from measurements [7].

Using Newton-Gauss type algorithms, the coefficients of the transfer function can be estimated by minimizing the following cost function

$$\arg \min_{c_p, \tilde{c}_p} \sum_{k=0}^{K} \left| H(s_k) - \frac{N(s_k)}{D(s_k)} \right|^2 \tag{2}$$

which is non-linear in terms of c_p and $\tilde{c}_p$ [8]. Unfortunately, such non-linear optimization techniques are computationally not always efficient, and the solutions may converge to local minima, even when Levenberg-Marquardt algorithms are used to extend the region of convergence [9][10]. A possible option to remove non-linearity, is to assume that the poles are known from a-priori knowledge [11]. In practice such information is often not available, and rough estimates of the pole-locations are rarely sufficient to capture all dynamics of the spectral response, especially if the behaviour of the transfer function isn't smooth.

Levi [12] suggested to minimize the following cost-function which is non-quadratic in the system parameters

$$\arg \min_{c_p, \tilde{c}_p} \sum_{k=0}^{K} |D(s_k)H(s_k) - N(s_k)|^2 \tag{3}$$

It should be noted that (2) and (3) are not equivalent in general, since the calculated transfer function will be biased by an unbalanced weighting. This often results in poor low-frequency fits, due to an undesired overemphasis of high-frequency errors. In this paper, the use of a Sanathanan-Koerner iteration [4] is proposed to alleviate the bias. Based on an initial (iteration step 0) or previous (iteration step $t - 1$) estimate of the poles, the model parameters of the next iteration step are calculated by minimizing the weighted linear cost function

$$\arg \min_{c_p, \tilde{c}_p} \left(\sum_{k=0}^{K} \frac{\left| D^{(t)}(s_k)H(s_k) - N^{(t)}(s_k) \right|^2}{\left| D^{(t-1)}(s_k) \right|^2} \right) \tag{4}$$

By analyzing the gradients of the error criterion, it can be shown that (2) is not equivalent to (4) either, even though the error criterion itself tends asymptotically to the fundamental least squares criterion [13]. In practice, however, this approach often gives favorable results for sufficiently high signal-to-noise ratios and sufficiently small modeling errors. The interested reader is hereby referred to an excellent survey [8].

3 Muntz-Laguerre basis functions

A straightforward way to calculate an orthonormal basis, is to apply a Gram-Schmidt procedure on a set of partial fractions [5][14]. Hence, orthonormal rational functions $\phi_p(s)$ are obtained, which are in fact linear combinations of the partial fractions, of the form

$$\phi_p(s) = \frac{Q_p(s)}{\prod_{j=1}^{p}(s + a_j)} \tag{5}$$

for $p = 1, ..., P$ and $Q_p(s)$ an arbitrary polynomial of order $p - 1$, such that

$$\langle \phi_m(s), \phi_n(s) \rangle = \delta_{mn} \tag{6}$$

for $1 \leq m, n \leq P$. If the inner product is defined as

$$\langle \phi_m(s), \phi_n(s) \rangle = \frac{1}{2\pi i} \int_{i\mathbb{R}} \phi_m(s)\phi_n^*(s)ds \tag{7}$$

then the $Q_p(s)$ polynomial can be determined by imposing the orthonormality conditions on the basis functions. The Gram-Schmidt procedure is done analytically and gives rise to the following closed form expression

$$\phi_p(s) = \frac{\kappa_p \sqrt{2\Re e(a_p)}}{s + a_p} \left(\prod_{j=1}^{p-1} \frac{s - a_j^*}{s + a_j} \right) \tag{8}$$

provided that the poles are stable, i.e. located in the left half of the complex plane. κ represents an arbitrary unimodular complex number. This basis originates from the discrete-time Takenaka-Malmquist basis [15][16], and

has later been transformed to the continuous time domain. It is a generalization of the Laguerre basis [17], where all poles $\{-a_p\}$ are the same real number, and the 2-parameter Kautz bases [18] where all poles $\{-a_p, -a_{p+1}\}$ are the same complex conjugate pair with $-a_p^* = -a_{p+1}$. A theoretical analysis of these basis functions is well-described in literature. The interested reader is referred to [19] which gives an excellent survey.

To make sure that the transfer function has real-valued coefficients, a linear combination of $\phi_p(s)$ and $\phi_{p+1}(s)$ is formed which can be made real-valued if the poles are real or occur in a complex conjugate pair. This way, two orthonormal functions of the following form are obtained

$$\phi_p(s) = \frac{\gamma_p(s-x)}{(s+a_p)(s+a_{p+1})}\left(\prod_{j=1}^{p-1}\frac{s-a_j^*}{s+a_j}\right) \tag{9}$$

$$\phi_{p+1}(s) = \frac{\gamma_{p+1}(s-y)}{(s+a_p)(s+a_{p+1})}\left(\prod_{j=1}^{p-1}\frac{s-a_j^*}{s+a_j}\right) \tag{10}$$

To satisfy the orthonormality, the indeterminates x and y are chosen to be $\sqrt{a_p a_{p+1}} = |a_p|$ and $-\sqrt{a_p a_{p+1}} = -|a_p|$ respectively, and γ_p and γ_{p+1} are set to $\sqrt{a_p + a_{p+1}} = \sqrt{2\Re e(a_p)}$. Note that this choice is not unique, and that other possibilities exist.

4   Iterative Pole-relocation process

Let's define $\phi_p(s,a)$ as the pth orthonormal rational basis function, based on a prescribed set of poles $a = \{-a_1, ..., -a_P\}$, and evaluated at complex frequency s. Then the numerator and denominator can be factorized as follows

$$N(s) = \sum_{p=1}^{P} c_p \phi_p(s,a) = \frac{\prod_{p=1}^{P-1}(s+z_{p,n})}{\prod_{p=1}^{P}(s+a_p)} \tag{11}$$

$$D(s) = \tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p \phi_p(s,a) = \frac{\prod_{p=1}^{P}(s+z_{p,d})}{\prod_{p=1}^{P}(s+a_p)} \tag{12}$$

and the transfer function $R(s)$ is obtained as

$$R(s) = \frac{N(s)}{D(s)} = \frac{\prod_{p=1}^{P-1}(s+z_{p,n})}{\prod_{p=1}^{P}(s+z_{p,d})} = \sum_{p=1}^{P}\alpha_p\phi_p(s,z_d) \tag{13}$$

After simplification, it is easy to see that the poles of the overall transfer function are in fact the zeros $-z_{p,d}$ of the denominator $D(s)$. Using the pole-relocation process, the zeros can be calculated and relocated in order to minimize the global fitting error. This goal is achieved by a sequential minimization of Levi's cost function and several Sanathanan-Koerner iterations (see below). Afterwards, one can calculate the model coefficients α of the transfer function as a linear problem.

4.1   Levi's estimator

In the first iteration step $(t = 0)$, Levi's estimator is applied to obtain a first guess of the denominator $(D^{(-1)}(s) = 1)$. The poles of the basis functions are chosen to be a set of prescribed poles a.

$$\arg\min_{\tilde{c}_p^{(t)}, c_p^{(t)}} \left(\sum_{k=0}^{K}\left|\frac{1}{D^{(t-1)}(s_k)}\right|^2 \left|D^{(t)}(s_k)H(s_k) - N^{(t)}(s_k)\right|^2\right) \tag{14}$$

$$= \arg\min_{\tilde{c}_p^{(0)}, c_p^{(0)}} \left(\sum_{k=0}^{K}\left|\left(\sum_{p=1}^{P}\tilde{c}_p^{(0)}\phi_p(s_k,a) + \tilde{c}_0^{(0)}\right)H(s_k) - \sum_{p=1}^{P}c_p^{(0)}\phi_p(s_k,a)\right|^2\right) \tag{15}$$

One coefficient of the rational function, e.g. $\tilde{c}_0^{(0)}$, can be fixed to unity, since numerator and denominator can be divided by the same complex value without loss of generality. Then, the identification problem reduces to solving the following set of least-squares equations for all complex frequencies s.

$$\sum_{p=1}^{P} c_p^{(0)} \phi_p(s,a) - \left(\sum_{p=1}^{P} \tilde{c}_p^{(0)} \phi_p(s,a) \right) H(s) = H(s) \tag{16}$$

After identification of the coefficients $c_p^{(0)}$ and $\tilde{c}_p^{(0)}$, the numerator $N^{(0)}(s)$ and denominator $D^{(0)}(s)$ are known (11-12). Since they share a common set of poles, the transfer function reduces to a strictly proper rational function after simplification. Clearly, the zeros $-z_{p,d}^{(0)}$ of $D^{(0)}(s)$ now become the transfer function poles. These poles can easily be calculated by constructing the minimal state-space realization (A,B,C,D) of $D^0(s)$, as described in [6], and by solving the following eigenvalue problem

$$-z_{p,d}^{(0)} = eig(A - BC) \tag{17}$$

4.2 Sanathanan-Koerner iteration

Now, the Sanathanan-Koerner linearization can be applied to relieve the bias, for iteration step $t = 1, ..., T$

$$\arg \min_{\tilde{c}_p^{(t)}, c_p^{(t)}} \left(\sum_{k=0}^{K} \left| \frac{1}{D^{(t-1)}(s_k)} \right|^2 \left| D^{(t)}(s_k) H(s_k) - N^{(t)}(s_k) \right|^2 \right) \tag{18}$$

$$= \arg \min_{z_{p,d}^{(t)}, z_{p,n}^{(t)}} \left(\sum_{k=0}^{K} \left| \frac{\prod_{p=1}^{P}(s_k + a_p)}{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t-1)})} \right|^2 \left| \left(\frac{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t)})}{\prod_{p=1}^{P}(s_k + a_p)} \right) H(s_k) - \frac{\prod_{p=1}^{P-1}(s_k + z_{p,n}^{(t)})}{\prod_{p=1}^{P}(s_k + a_p)} \right|^2 \right) \tag{19}$$

$$= \arg \min_{z_{p,d}^{(t)}, z_{p,n}^{(t)}} \left(\sum_{k=0}^{K} \left| \left(\frac{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t)})}{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t-1)})} \right) H(s_k) - \frac{\prod_{p=1}^{P-1}(s_k + z_{p,n}^{(t)})}{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t-1)})} \right|^2 \right) \tag{20}$$

$$= \arg \min_{\tilde{d}_p^{(t)}, d_p^{(t)}} \left(\sum_{k=0}^{K} \left| \left(\sum_{p=1}^{P} \tilde{d}_p^{(t)} \phi_p(s_k, z_d^{(t-1)}) + \tilde{d}_0^{(t)} \right) H(s_k) - \sum_{p=1}^{P} d_p^{(t)} \phi_p(s_k, z_d^{(t-1)}) \right|^2 \right) \tag{21}$$

Again, this minimization problem reduces to solving the following set of linear least-squares equations for all complex frequencies s.

$$\sum_{p=1}^{P} d_p^{(t)} \phi_p(s, z_d^{(t-1)}) - \left(\sum_{p=1}^{P} \tilde{d}_p^{(t)} \phi_p(s, z_d^{(t-1)}) \right) H(s) = H(s) \tag{22}$$

Clearly, (22) resembles (16), except that the set of prescribed poles is replaced by the previously identified poles. This is caused by the fact that now the coefficients $d_p^{(t)}$ and $\tilde{d}_p^{(t)}$ of $N^{(t)}(s)/D^{(t-1)}(s)$ and $D^{(t)}(s)/D^{(t-1)}(s)$ are estimated instead of the coefficients $c_p^{(t)}$ and $\tilde{c}_p^{(t)}$ of $N^{(t)}(s)$ and $D^{(t)}(s)$. Note however that the zeros of $D^{(t)}(s)$ and $D^{(t)}(s)/D^{(t-1)}(s)$ are the same, so the relocated poles of the transfer function can be calculating by solving the eigenvalue problem, based on the minimal state-space realization of $D^{(t)}(s)/D^{(t-1)}(s)$. Since the poles of the transfer function are relocated in each process, one can consider the Sanathanan-Koerner iteration as a *pole-relocation process*, even though the poles of the basis functions remain unchanged. Once the Sanathanan-Koerner iteration has converged, the final poles of the transfer function are *known*, and the non-linear identification problem reduces to a *linear* problem. Unstable poles can be flipped into the left half plane before each iteration.

5 Estimation of model coefficients

After convergence, one can simply rewrite the transfer function as a linear combination of orthonormal rational functions which are based on the identified poles.

$$R(s) = \sum_{p=1}^{P} \alpha_p^{(T)} \phi_p(s, z_d^{(T-1)}) \tag{23}$$

The coefficients $\alpha_p^{(T)}$ of $R(s)$ can be estimated by solving the following linear problem

$$\arg\min_{\alpha_p^{(T)}} \sum_{k=0}^{K} \left| H(s_k) - \sum_{p=1}^{P} \alpha_p^{(T)} \phi_p(s_k, z_d^{(T-1)}) \right|^2 \tag{24}$$

If unstable poles are allowed, one can identify the residues in a basis of partial fractions [20].

$$\arg\min_{\alpha_p^{(T)}} \sum_{k=0}^{K} \left| H(s_k) - \sum_{p=1}^{P} \frac{\alpha_p^{(T)}}{s_k + z_{p,d}^{(T-1)}} \right|^2 \tag{25}$$

6   Example

6.1   Multipole Filter

The reflection coefficient S_{11} of a two-port dispersive multipole microstrip filter is modeled using the proposed modeling and identification technique, based on full-wave EM simulations [21].

Figure 1 shows the magnitude of the spectral response over the frequency range of interest [0.1 GHz - 30 GHz]. In this example, all frequencies are scaled by 10^9.
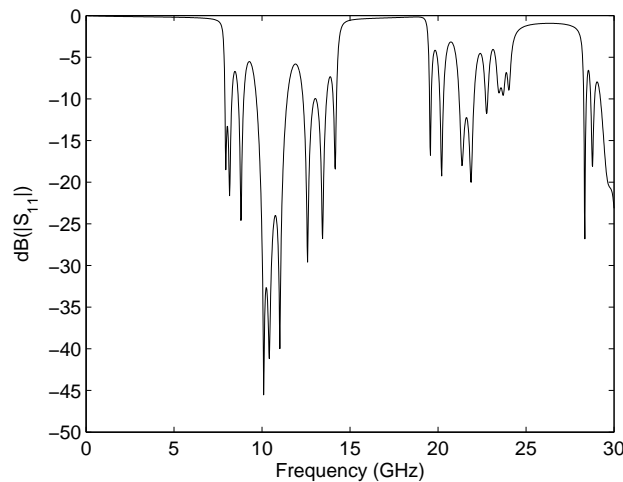
Figure 1: Magnitude Multipole Filter (S_{11}).

First, a prescribed set of 70 complex conjugate starting poles is chosen as was proposed in [20]

$$-a_p = -\alpha + \beta i, -a_{p+1} = -\alpha - \beta i \tag{26}$$
$$\alpha = \beta/100 \tag{27}$$

with imaginary parts β covering the frequency range of interest. Other prescribed pole-location schemes are also possible, however they often require more iterations before the poles are converged. For more details, the reader is referred to [20]. The error criterion is defined as

$$Error = \max(dB(|S_{11} - S_{fit}|)) \tag{28}$$

The weighted linear cost function (3) is minimized by solving (16) using the orthonormal rational basis functions (8)(9)(10) and an estimate for the residues c_p and $\tilde{c}_p$ is obtained. Using the residues $\tilde{c}_p$ and the poles $-a_p$, the minimal state-space realization (A, B, C, D) of the denominator $D(s)$ is calculated. From this state-space model, the poles of the transfer function are calculated by solving the eigenvalues of $A - BC$. Based on these relocated poles, the Sanathanan-Koerner iteration (4) can be applied by solving (22) iteratively. In each iteration, a relocated set of poles is calculated and this process is repeated until the poles are converged to their optimal location. For

this example, the algorithm was applied for 7 iterations. Once the converged poles are known, the residues of the transfer function α_p can be estimated by solving (24) or (25).

Figure 2 shows the relocation of the poles for all consecutive iterations, and the corresponding fitting accuracy.

The condition number of the pole identification problem and the overall modeling accuracy of the consecutive iterative steps are shown in Table 1. In each iteration, a well-conditioned system of equations is solved. Quite often, the accuracy of the transfer function improves during the iterative process, as the poles get relocated to more optimal locations in the complex plane. The iterative process is stopped when the overall accuracy level (-71 dB) reaches the numerical noise level of the simulator [21].

Iter	Cond.Nr. Pole-Ident	Accuracy fit
1	1.8762×10^6	-26.5689 dB
2	6.4074×10^5	-38.1430 dB
3	7.1195×10^5	-38.7002 dB
4	6.3264×10^5	-44.6632 dB
5	5.4745×10^5	-63.9071 dB
6	6.1792×10^5	-69.5781 dB
7	3.2272×10^5	-71.7025 dB

Table 1: Conditioning of pole-identification & Accuracy per iteration.

7   Conclusions

A computationally efficient and reliable identification method for accurate transfer function synthesis is described. The method starts from a prescribed set of starting poles, and relocates them using a Sanathanan-Koerner iteration. Orthonormal rational functions are used to improve the numerical conditioning of the system equations. The main functionality of the method is illustrated by an example.
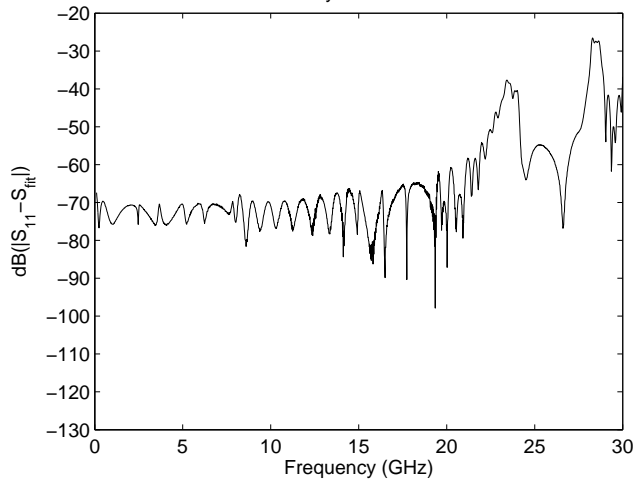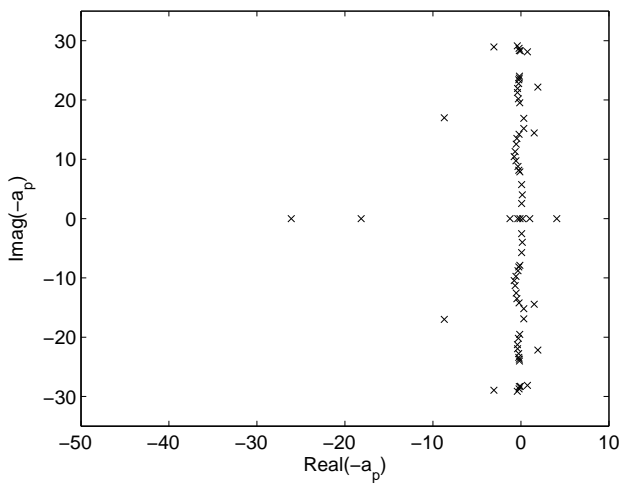
8   Acknowledgements

Starting poles
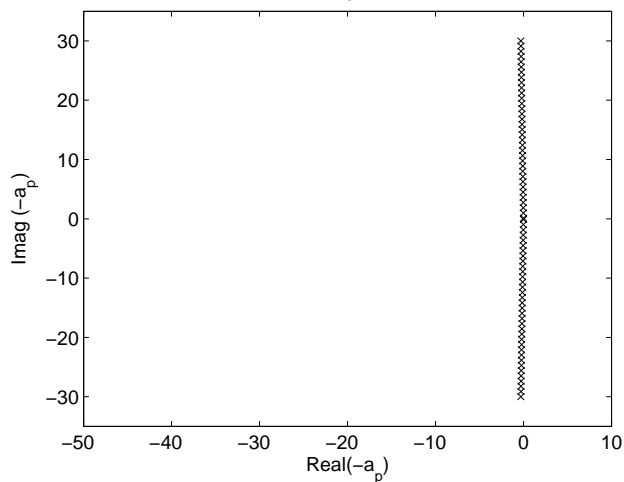
Iteration 1
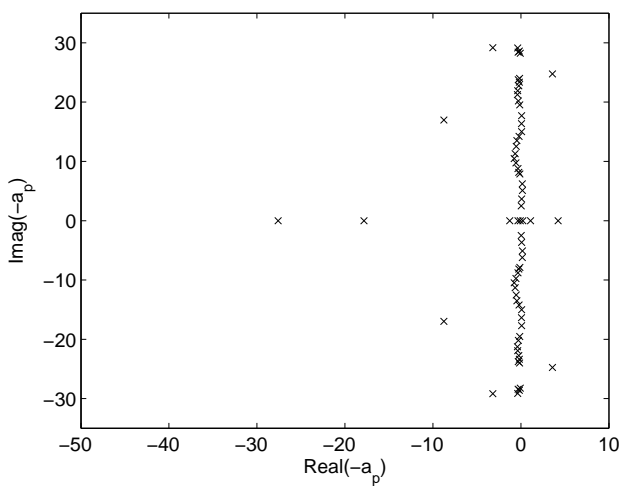
Accuracy : −26.5689 dB

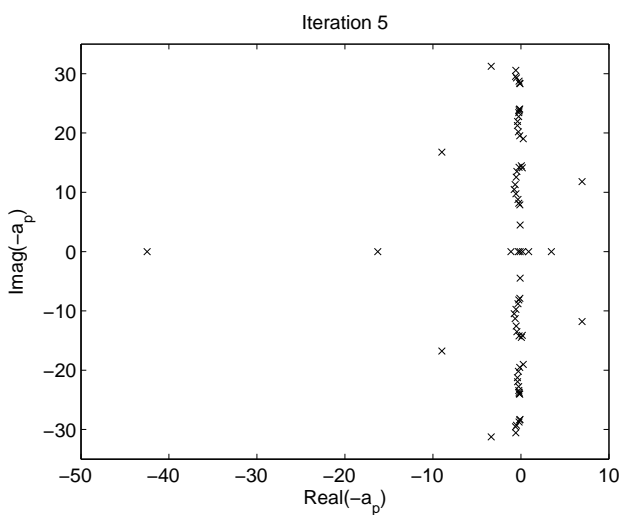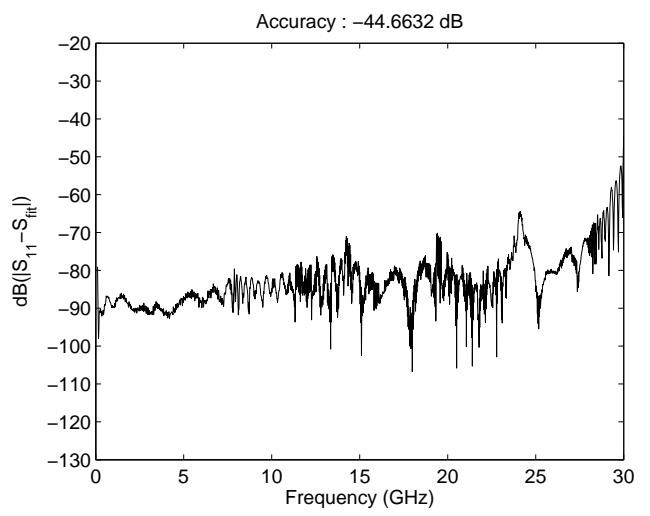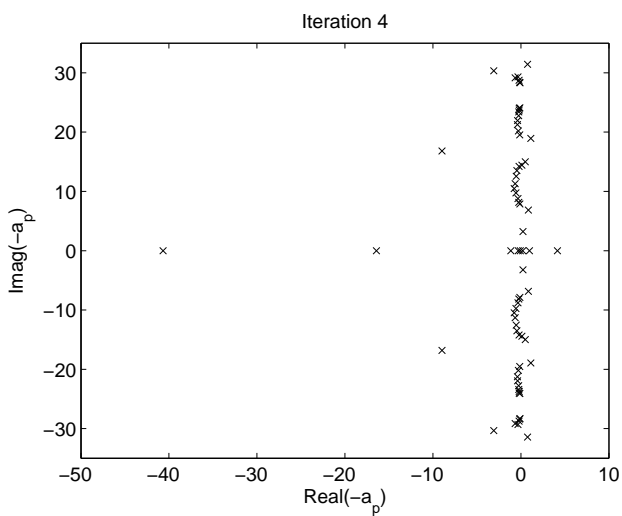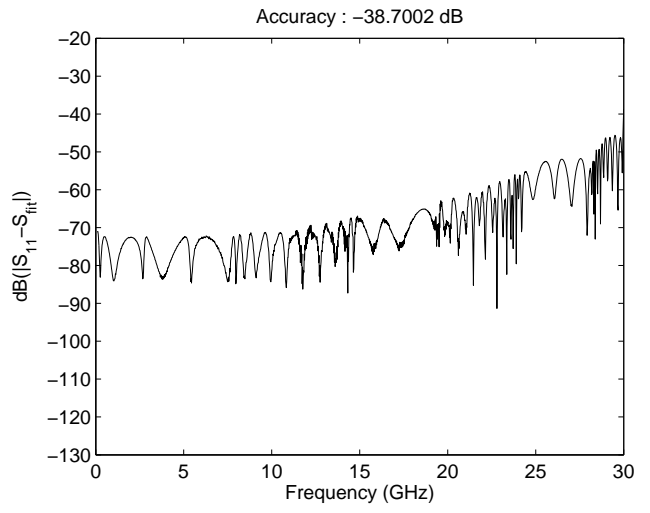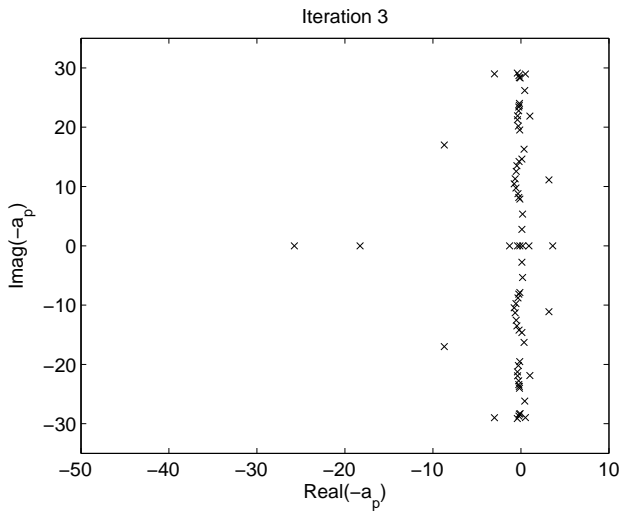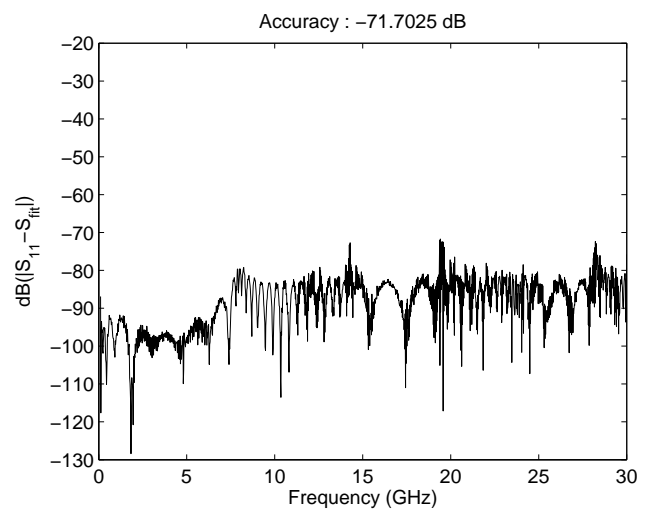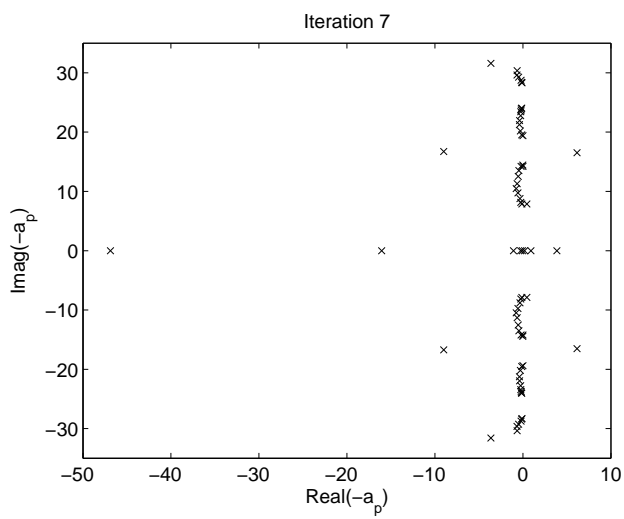## Iteration 2
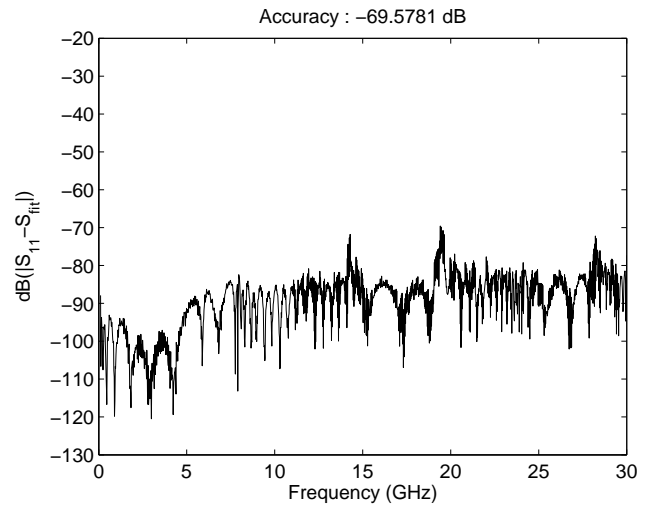
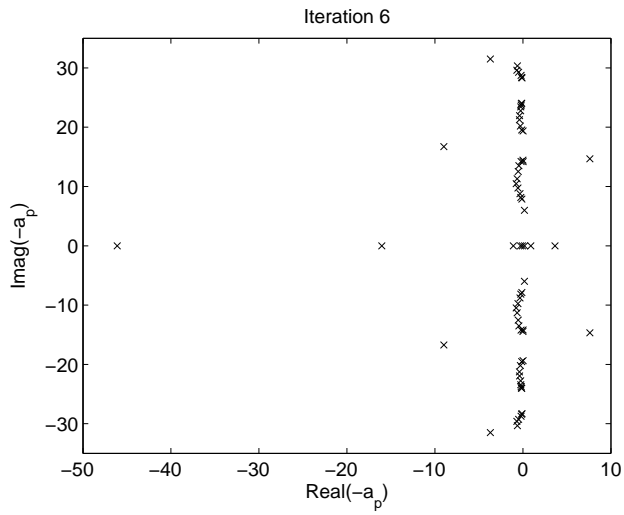## Accuracy : −38.1430 dB

1079
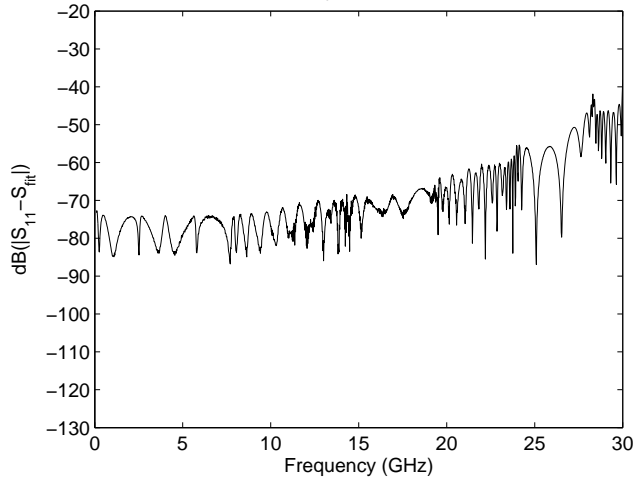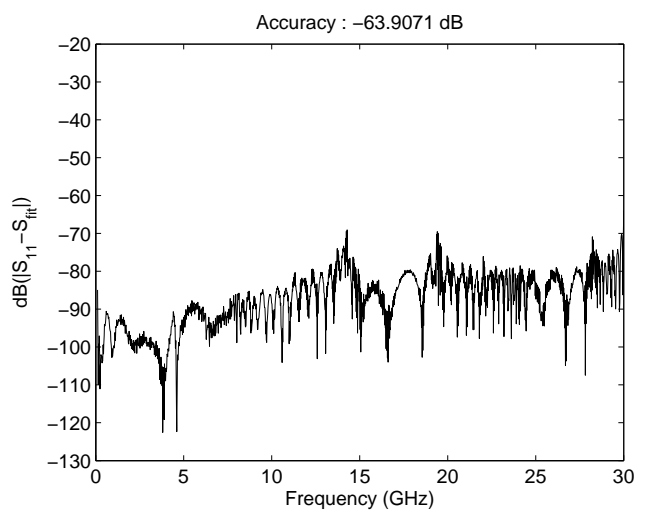
Figure 2: Iterative pole relocation process : identified poles (left) and fitting accuracy (right)

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